SPHERICAL TRIGONOMETRY

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SPHERICAL TRIGONOMETRY

FOR

COLLEGES AND SECONDARY SCHOOLS

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PREFACE.

THIS book contains little more than what is required for the solution of spherical triangles and related simple practical problems. The articles on spherical geometry are necessary for those who have not already studied that subject; for others, they provide a useful review. More than usual attention has been given to the measurement of solid angles. The explanations in connection with the astronomical problems are somewhat fuller than is customary in elementary text-books on spherical trigonometry.

I am indebted to Mr. W. B. Fite, Ph.B., Fellow in Mathematics at Cornell University, for his kind assistance in reading the proof-sheets; and to Mr. A. T. Bruegel, M.M.E., of the Pratt Institute, Brooklyn, N.Y., for the pleasing character of the diagrams.

D. A. MURRAY.

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CONTENTS.

CHAPTER I.

Review of Solid and Spherical Geometry.

CHAPTER II.

Right-angled Spherical Triangles.

CHAPTER III.

Relations between the Sides and Angles of Spherical Triangles.

CHAPTER IV.

Solution of Triangles.

CHAPTER V.

Circles connected with Spherical Triangles.

CHAPTER VI.

Areas and Volumes connected with Spheres.

CHAPTER VII.

PRACTICAL APPLICATIONS.

APPLICATIONS TO ASTRONOMY.

APPENDIX.

SPHERICAL TRIGONOMETRY.

CHAPTER I.

REVIEW OF SOLID AND SPHERICAL GEOMETRY.

On beginning the study of spherical trigonometry it is advisable to recall to mind or learn some of the definitions and propositions of solid geometry. A clear and vivid conception of the principal properties of the sphere is especially necessary. The definitions and theorems which will be used frequently in the following pages, are quoted in this chapter.*

Planes and Lines in Space. Diedral Angles. Solid Angles.

1. ***a.*** Two planes which are not parallel intersect in a straight line. (Euc. XI. 3.)

b. The angle which one of two planes makes with the other is called a diedral angle. Thus, in Fig. 1, the two planes BD and
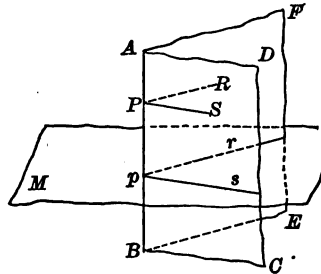
FIG. 1

* As far as possible, references are made to the text of Euclid; since, of the numerous geometrical text-books in English-speaking countries, his work is the one which is most largely used. Those who use a text-book other than Euclid's can substitute the appropriate references.

AE intersect in the straight line *AB*, and form the diedral angle *FABC*.

c. The planes *AE* and *AC* are called the *faces*, and the line *AB* is called the *edge*, of the diedral angle. The faces are unlimited in extent. The magnitude of the diedral angle depends, not upon the extent of its faces, but only upon their relative position. (Just as the magnitude of a plane angle depends, not upon the lengths of its boundary lines, but upon their relative position.)

d. If *PR* be drawn perpendicular to *AB* in the plane *AE*, and *PS* be drawn perpendicular to *AB* in the plane *AC*, the angle *RPS* is called *the plane angle of the diedral angle.*

e. If a plane is drawn perpendicular to the edge of a diedral angle, the intersections of this plane with the faces of the diedral angle form the plane angle of the diedral angle. (See Euc. XI. 4.) Thus, if the plane *M* be passed through *p* perpendicular to *AB*, the intersections, *pr*, *ps*, of the plane *M* and the planes *AE*, *AC*, form the angle *rps* which is the plane angle of *FABC*.

f. All plane angles of the same diedral angle are equal. (See Euc. XI. 10.) Hence, *the plane angle can be taken as the measure of the diedral angle.*

2. *a.* If a straight line be at right angles to a plane, every plane which passes through the line is at right angles to that plane. (Euc. XI. 18.)

b. If two planes which cut one another be each of them perpendicular to a third plane, their common section is perpendicular to the same plane. (Euc. XI. 19.)

3. *a.* When three or more planes meet in a common point, they are said to form a *solid angle*, or a *polyedral angle*, at that point.

The point in which the planes meet is called the *vertex* of the solid angle; the intersections of the planes are called its *edges;* the portions of the planes between the edges are called its *faces;* the plane angles formed by the edges are called its *face angles;* and the diedral angles formed at the edges by the planes are called the *diedral angles* (or the *edge angles*) of the solid angle.

Thus, in Fig. 2, for the solid angle formed at *S*: the vertex is *S*; *SB, SC, SD, SE,* are the edges; *BSE, ESD,* etc., are the faces; the face angles are the angles *BSE, ESD, DSC, CSB*; the diedral (or edge) angles are *BESD, EDSC,* etc.
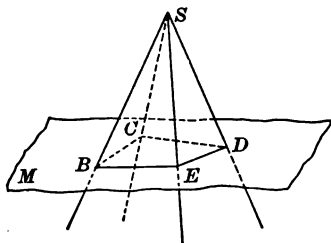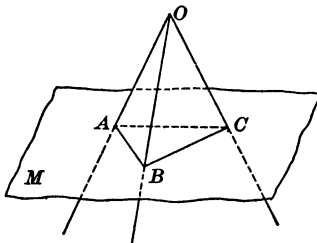
FIG. 2 FIG. 3

b. A solid angle with three faces is called a *triedral* angle. Thus, the solid angle at *O* (Fig. 3) is a triedral angle.

(The measurement of solid angles is discussed in Art. 61. The magnitude of the solid angle in nowise depends upon the lengths of its edges.)

4. ***a.*** The sum of any two face angles of a triedral angle is greater than the third. (See Euc. XI. 20.)

b. The sum of the face angles of any solid angle is less than four right angles (Euc. XI. 21). (This is true, in general, only when the polygon, say *BEDC* (Fig. 2), formed by the intersections of the faces with a cutting plane *M*, does not have a re-entrant angle; in other words, when the polygon *BEDC* is convex.

Geometry of the Sphere.

For the benefit of those who have not studied the geometry of the sphere, proofs of a few of its propositions are either outlined, or given in detail. Some propositions can be proved very easily; hence, only their enunciations are given. Other properties of the sphere will be proved when they are required. (See Arts. 53, 54, 57, 62, 65.) The use of a globe on which figures can be drawn, will be of great assistance to the student. If such a globe is not at hand, a terrestrial or celestial globe can afford some service.

5. The sphere and its plane sections.

a. **Definitions.** A *spherical surface* is a surface all points of which are equidistant from a point called the *centre*. A *sphere* is a solid bounded by a spherical surface. The surface of a sphere can be generated by the revolution of a semicircle about its diameter. A *radius* of a sphere is a straight line joining the centre to any point on the surface. According to the definition of a sphere, all the radii of a sphere are equal. A *diameter* of a sphere is a straight line passing through the centre and terminated at both ends by the surface. A *plane section of a sphere* is a figure whose boundary is the intersection of a plane and the surface of the sphere.

b. **Proposition.** *The boundary of every plane section of a sphere is a circle.*

Let the sphere whose centre is at O be cut by a plane in the section ABD; then ABD is a circle. Through O draw OC perpendicular to the plane ABD. Let A and B be any two points in the boundary of the section ABD. Draw OA, OB, CA, and CB. In the two triangles OCA and OCB, the angles at C are equal (both being right angles), the side OC is common, and the side OA is equal to the side OB, since both are radii of the sphere. Hence the triangles are equal in every respect, and CA is equal to CB. But A and B are any two points on the boundary of the section; hence all points on the boundary are equidistant from C. Therefore ABD is a circle whose centre is at C, the foot of the perpendicular let fall from the centre O to the cutting plane ABD.
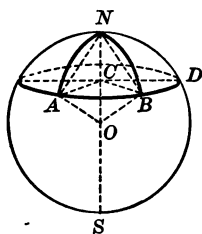
FIG. 4

6. Great and small circles on a sphere.

a. **Definitions.** The section in which a sphere is cut by a plane is called a *Great Circle* when the plane passes through the centre of the sphere; the section is called a *Small Circle* when the cutting plane does not pass through the centre of the sphere. Thus, on a terrestrial globe the meridians and equator are great circles; the parallels of latitude are small circles. The *Axis* of a circle of

a sphere is the diameter of the sphere perpendicular to the plane of the circle; the extremities of the axis are called the *Poles* of the circle and any of its arcs. Thus, in Fig. 4, Art. 5, N and S are the poles of the circle ABD and of the arcs AB and BD. It is obvious that all circles made by the intersections of parallel planes with a sphere have the same axis and poles. For instance, all parallels of latitude have the same axis and poles, namely, the polar axis of the earth and the North and South Poles.

***b.* Propositions relating to great circles.**

Every great circle bisects the surface of the sphere; *e.g.* the equator bisects the surface of a terrestrial globe.

Any two great circles bisect each other; *e.g.* the meridians bisect one another at the poles. All great circles of a sphere are equal; since their radii are radii of the sphere.

A great circle can be passed through any two points on a sphere; since a plane can be made to pass through these two points and the centre of the sphere, and this plane intersects the surface of a sphere in a great circle. In general, only one great circle can be drawn through two points on a sphere, since these points and the centre determine a plane; but, when the two given points are at the ends of a diameter an infinite number of great circles can be drawn through them; *e.g.* the meridians passing through the North and South Poles.

***c.* Definitions.** By **distance between two points on a sphere** is meant the shorter arc of the great circle passing through them. It is shown in Art. 20 that this arc is the *shortest line* that can be drawn on the surface of the sphere from the one point to the other. For example, the arc NA in Fig. 4 measures the distance between the points N and A. [Ex. Distance between N and S?]

NOTE. The theorem in Art. 20 can be shown *mechanically* by taking two points on a parallel of latitude on a globe and letting a string be stretched taut from one point to the other. The string will not lie on the parallel, but will evidently be in a plane which passes through the centre of the sphere. If the two points be on a meridian, the stretched string will lie on the meridian.

By **angular distance** between two points on a sphere is meant the angle subtended at the centre of the sphere by the arc joining the given points. Thus in Fig. 4 the angle NOA is the angular distance of A from N.

d. Propositions and definitions relating to small and great circles. In Fig. 4 all the arcs of great circles, as *NA, NB, ND,* drawn from points on the circle *ABD* to the pole *N,* are equal. Thus the arcs of meridians on a terrestrial globe drawn from a parallel of latitude to the North Pole are equal. The chords *NA, NB, ND,* are all equal; the angles *AON, BON, DON,* are likewise equal. It thus appears that all points in the circumference of a circle on a sphere are equally distant from a pole of the circle, whether the distance be measured by the arc of a great circle joining one of the points and the pole, or by the straight line joining the point and the pole, or by the angle which such an arc or chord subtends at the centre of the sphere.

Definitions. The last mentioned angle is called the *angular radius* of the circle. The angular radius of a great circle is evidently a right angle. The *polar distance* of a circle on a sphere is its distance from its pole, the distance being measured along an arc of a great circle passing through the pole. Thus the north polar distance of a parallel of latitude is its distance from the North Pole measured along a meridian. The term *quadrant,* when used in connection with a sphere, usually means an arc equal in length to one-fourth of a great circle. The polar distance of each point on a great circle is evidently a quadrant; *e.g.* a point on the equator is at a quadrant's distance from the North or South Pole. Points on a great circle are equidistant from both its poles. The polar distance of a circle may be called the *radius* of the circle.

7. To draw circles upon the surface of a sphere about a given point as pole.

(*a*) *With a pair of compasses.* Open the compasses until the distance between the points of the compasses is equal to the chord of the polar distance (or, what is the same thing, the chord subtended by the angular radius) of the required circle. Then, one point being placed and kept fixed at the pole, the other can describe the circle.

(*b*) *With a string.* Take a string equal in length to the polar distance of the required circle. If the string be kept stretched

taut, and one end be fixed at the pole while the other end moves on the sphere, the required circle will be described.

In order to describe a great circle the polar distance must be taken equal to a quadrant of the sphere.

8. Proposition. *If a point on the surface of a sphere lies at a quadrant's distance from each of two points, it is the pole of the great circle passing through these points.*

If the point P be at a quadrant's distance from each of the points A and B, then P is the pole of the great circle passing through A and B. Let O be the centre of the sphere, and draw OA, OB, OP. Since PA and PB are quadrants, the angles POA and POB are right angles. Hence PO is perpendicular to the plane AOB (Euc. XI. 4); therefore P is the pole of the great circle ABL.
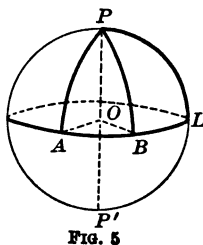
FIG. 5

9. Problem. *Through two given points to draw an arc of a great circle.* About each point as a pole draw a great circle (Art. 7). The two points of intersection of the great circles thus drawn are each at a quadrant's distance from the two given points; and hence, by Art. 8, are the poles of the great circle through the two given points. Accordingly, the required arc will be obtained by describing a great circle about either of these poles.

NOTE. If the two given points are diametrically opposite, an infinite number of great circles can be drawn through them. (Art. 6. *b*.)

10. Lines and planes which are tangent to a sphere.

***a*. Definitions.** A straight line or a plane is said to be *tangent* to a sphere when it has but one point in common with the surface of the sphere. The common point is called the *point of contact* or *point of tangency.*
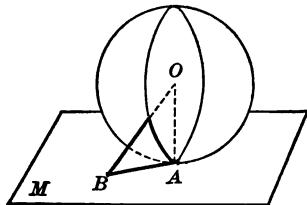
FIG. 6

b. Propositions. (See Fig. 6.)

A plane or a line perpendicular to a radius at its extremity is tangent to the sphere. [Suggestion for proof: The perpendicular is the shortest line that can be drawn from a point to a plane.]

A tangent to an arc of a great circle at any point of the arc is perpendicular to the radius (of the sphere) drawn to the point.

11. On spherical angles.

***a.* Definitions.** *The angle made by any two curves* meeting in a common point is the angle formed by the two tangents to the curves at that point. Thus in Fig. 7, the angle made by the curves C_1 and C_2 at the point P, is the angle T_1PT_2 between the tangents to C_1 and C_2 at P. (This definition applies to all curves, whether they are in the same plane or not.)
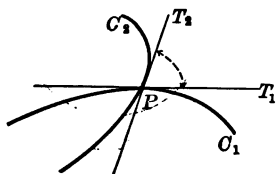
FIG. 7

A spherical angle is the angle formed by two intersecting arcs of great circles on the surface of a sphere. Thus the angle formed by the arcs CA and CB (Fig. 8) is a spherical angle. This angle is the angle ECD between the tangents CE and CD. But ECD is the plane angle of the diedral angle between the planes COA and COB which are the planes of the arcs CA and CB. Thus *the spherical angle is equal to the diedral angle of the planes of the arcs forming the angle.*
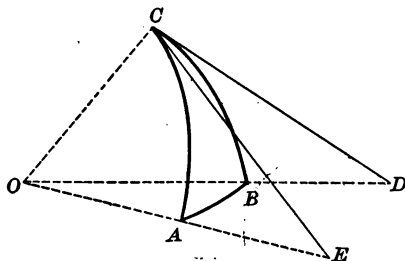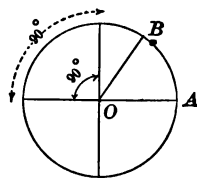
FIG. 8 FIG. 9

***b.* Propositions.** (1) If two arcs of great circles intersect, the opposite vertical angles thus formed are equal. Thus in Fig. 49, Art. 57, the angles BAC and $B'AC'$ are equal.

(2) If one arc of a great circle meets another arc of a great circle, the sum of the adjacent spherical angles is equal to two right angles. Thus in Fig. 49, $CAB + CAB' = 2$ right angles.

NOTE. It is shown in plane geometry that angles at the centre of a circle are proportional to their intercepted arcs; hence, the angles can be measured by the arcs. Accordingly, if each right angle at the centre of a circle (Fig. 9) be divided into 90 equal parts called degrees, and the circle be divided into 360 equal parts, also called degrees, then the number of degrees (of angle) in any angle AOB is equal to the number of degrees (of arc) in AB, the arc subtended by AOB. [When it is necessary to distinguish between *degrees of angle* and *degrees of arc*, the former may be called *angular degrees;* and the latter *arcual degrees.*]

c. **Proposition.** *A spherical angle is measured by the arc of a great circle described with its vertex as a pole and included between its boundary arcs, produced if necessary:*

Let ABC and $AB'C$ be two intersecting arcs of great circles on the sphere S whose centre is at O. Pass the plane BOB' through O perpendicular to AC, and let this plane intersect the planes ABC and $AB'C$ in the radii OB and OB', and intersect the sphere in the great circle $B'BL$. From the construction, A is the pole of the great circle $B'BL$. By Art. 1. *e.* BOB' is the plane angle of the diedral angle $BACB'$, and, accordingly (Art. 11. *a*), is equal to the spherical angle BAB'. Now, by the preceding note, the number of degrees in the arc BB' is equal to the number of degrees in the angle BOB'. Hence, the number of degrees in the arc BB' is equal to the number of degrees in the angle BAB'. In other words, the spherical angle BAB' is measured by the arc BB' of which A is the pole.
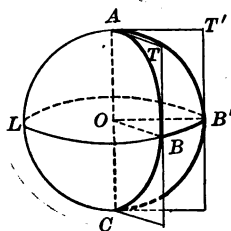
FIG. 10

This can be illustrated on a terrestrial globe. For instance, the angle at the North Pole between the meridians of Paris and New York is 76° 2′ 25.5″; and this is the number of degrees of arc intercepted by these meridians on the equator.

d. The great circles drawn through any point on a sphere are perpendicular to the great circle of which the point is the pole.

For instance, the meridians of longitude cross the equator at right angles.

e. The distance of any point on the surface of a sphere, from a circle traced thereon, is measured by the shorter arc of a great circle passing through the point and perpendicular to the given circle; that is, by the shorter arc of the great circle passing through the given point and the pole of the given circle. For example, on a globe the latitude of any place (*i.e.* its distance in degrees from the equator) is measured by the arc of the meridian intercepted between the place and the equator.

N.B. When an *arc on a sphere* is referred to, an arc of a great circle is meant, unless expressly stated otherwise.

ON SPHERICAL TRIANGLES.

12. Definitions. A *spherical polygon* is a portion of the surface of a sphere bounded by three or more arcs of great circles. The bounding arcs are the *sides* of the polygon; the points of intersection of the sides are the *vertices* of the polygon, and the angles which the sides make with one another are the *angles* of the polygon. A *diagonal* of a spherical polygon is an arc of a great circle joining any two vertices which are not consecutive.
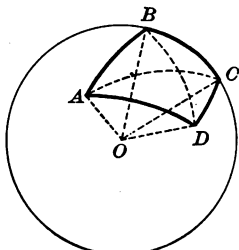
FIG. 11

A *spherical triangle* is a spherical polygon of three sides.

Thus, in Fig. 11, *ABCD* is a spherical polygon; its sides are *AB, BC, CD, DA*; its angles are *ABC, BCD, CDA, DAB*; its diagonals are *BD* and *AC*; *ADC* and *ABC* are spherical triangles. Since the sides of a spherical polygon are arcs of great circles, their magnitudes are expressed in degrees.* The lengths of the sides can be calculated in terms of *linear* units when the radius of the sphere is known.

A spherical triangle is *right-angled, oblique, scalene, isosceles,* or *equilateral,* in the same cases as a plane triangle. The notation

* The reason for expressing the sides of spherical polygons in degrees is considered more fully in Art. 14.

adopted in discussing the plane triangle will be used for the spherical triangle; namely, the triangle will be denoted by ABC, and the sides opposite the angles A, B, C, will be denoted by a, b, c, respectively.

Two spherical polygons are **equal** if they can be applied one to the other so as to coincide. They are said to be **symmetrical** when the sides and angles of the one are respectively equal to the sides and angles of the other, but arranged in *the reverse order.*
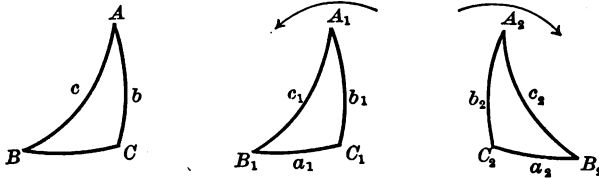
FIG. 12

Thus, the spherical triangles ABC and $A_1B_1C_1$ (Fig. 12) are *equal* if they can be brought into coincidence, say, by sliding one of them, as ABC, over the surface of the sphere until it exactly covers the surface $A_1B_1C_1$. Accordingly, it is evident that if these triangles are equal, the angles A, B, C, are respectively equal to the angles A_1, B_1, C_1, and the sides a, b, c, are respectively equal to the sides a_1, b_1, c_1.* On the other hand, the triangles ABC and $A_2B_2C_2$ are *symmetrical* if the angles A, B, C, are respectively equal to the angles A_2, B_2, C_2, and the sides a, b, c, to the sides a_2, b_2, c_2. In this case, the triangle ABC cannot be brought into coincidence with $A_2B_2C_2$ by a sliding motion over the surface of the sphere.

NOTE 1. Two symmetrical spherical triangles can be brought into coincidence if the surface be covered very thinly with some flexible material. For then ABC can be lifted up, turned over, and the surface bent (or made to 'spring back') in the opposite direction; after this treatment, ABC can be made to coincide with $A_2B_2C_2$.

NOTE 2. The meaning of the phrase *reverse order* can be seen clearly on considering the triangles $A_1B_1C_1$ and $A_2B_2C_2$ above. In $A_1B_1C_1$, on

* Some of the sets of *minimum* conditions necessary for equality of spherical triangles are stated in Art. 13.

going from A_1 to B_1, thence to C_1, and thence to A_1, one goes around any point within the triangle in a *counter-clockwise* direction. In $A_2B_2C_2$, on the other hand, on taking the respective equal angles in the same order as before, that is, on going from A_2 to B_2, thence to C_2, and thence to A_2, one goes round any point within the triangle $A_2B_2C_2$ in a *clockwise* direction. The directions are indicated by the arrows.

13. Propositions. (1) *Two spherical triangles which are on the same sphere, or on equal spheres, and whose parts are in the same order* (as ABC and $A_1B_1C_1$, Fig. 12) *are* **equal** *under the same conditions as plane triangles,* viz.:

(*a*) When two sides and their included angle in the one triangle are respectively equal to two sides and their included angle in the other;

(*b*) When a side and its two adjacent angles in the one triangle are respectively equal to a side and its two adjacent angles in the other;

(*c*) When the three sides of the one triangle are respectively equal to the three sides of the other.

[SUGGESTION FOR PROOFS. Equality can be shown by the same methods as in plane geometry.]

(2) *Two spherical triangles which are on the same sphere, or on equal spheres, and whose parts are in the reverse order* (as ABC and $A_2B_2C_2$, Fig. 12), are **symmetrical** under the conditions (*a*), (*b*), (*c*), above.

[SUGGESTIONS FOR PROOF. Construct* a triangle $A_1B_1C_1$ which is symmetrical to $A_2B_2C_2$. Under the given conditions, according to the preceding proposition, ABC and $A_1B_1C_1$ have all their parts respectively equal, and hence ABC and $A_2B_2C_2$ have all their parts respectively equal, and are accordingly symmetrical.]

On a plane two triangles may have three angles of the one respectively equal to three angles of the other and yet not be equal. On the other hand, as will be made apparent in Arts. 16, 24:

(3) *On the same sphere, or on equal spheres, two triangles which have three angles of the one respectively equal to three angles of the other, are either equal or symmetrical.*

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* For the construction of spherical triangles under various conditions, see Art. 24.

14. Correspondence between the face angles and the diedral angles of a triedral angle on the one hand, and the sides and angles of a spherical triangle on the other.

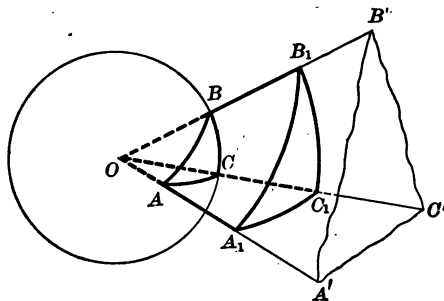



FIG. 13

Take any triedral angle $O\text{–}A'B'C'$; let a sphere of any radius, OA say, be described about O as centre; and let the intersections of this sphere with the faces $OA'B'$, $OB'C'$, and $OC'A'$, be the arcs AB, BC, and CA respectively. The sides of the spherical triangle ABC, namely, AB, BC, CA, measure the face angles, AOB, BOC, COA, of the solid angle $O\text{–}A'B'C'$ (Art. 11. *b*, Note). By Art. 11 the angles CAB, ABC, BCA, of the spherical triangle ABC are the diedral angles between the planes of the sides, that is, the diedral angles of the solid angle $O\text{–}A'B'C'$.

Hence, to find the relations existing between the face angles and the edge angles of a triedral angle, is the same thing as to find the relations between the sides and angles of the spherical triangle, intercepted by the faces, upon the surface of any sphere whose centre is at the vertex of the triedral angle.

NOTE 1. The number of degrees in the intercepted arcs does not depend upon the radius of the sphere. Thus, in Fig. 13, if a sphere is described with a radius OA_1, about O as a centre, the number of degrees in the intercepted arc A_1B_1 is the same as the number of degrees in the intercepted arc AB, for each number is the same as the number of degrees in the angle $A'OB'$.

Since the face angles and diedral angles of a triedral angle are not altered by varying the radius of the sphere, the relations between the sides and angles of the corresponding spherical triangle are independent of the length of the radius.

NOTE 2. Since the side of a spherical triangle measures the angle subtended by it at the centre, the side is measured in degrees or radians. (See Art. 12.) By "sin AB," for example, is meant the sine of the angle AOB, subtended by AB at the centre O.

NOTE 3. A three-sided spherical figure, one or more of whose sides is not an arc of a great circle, is not regarded as a spherical triangle. For example, the figure bounded by an arc of a parallel of latitude and the arcs of two meridians does not correspond to a triedral angle at the centre of the sphere, and is not a spherical triangle as defined in Art. 12.

NOTE 4. A triedral angle, and its corresponding spherical triangle, can be easily constructed. From stiff cardboard cut out a circular sector having any arc between 0° and 360°. On this sector draw any two radii, taking care, however, that no one of the three sectors thus formed shall be greater than the sum of the other two. Along these radii cut the cardboard partly through. Bend the two outer sectors over until their edges meet; a figure like O-ABC (Fig. 13) will be obtained. (Find what happens if the above precaution in drawing the radii is not taken.)

This perfect correspondence between the sides and angles of a spherical triangle on the one hand, and the face angles and diedral angles of the solid angle subtended at the centre of the sphere by the triangle on the other hand, **is very important,** both for the deduction of the relations between these sides and angles and for the solution of practical problems. This correspondence holds in the case of any spherical polygon and the solid angle subtended by it at the centre of the sphere. (The student may inspect Fig. 11.) Hence, *from any property of polyedral angles an analogous property of spherical polygons can be inferred, and vice versa.*

15. Propositions. (1) *Any side of a spherical triangle is less than the sum of the other two sides.* This follows from Arts. 14 and 4. *a*.

COR. Any side of a spherical polygon is less than the sum of the remaining sides.

(2) *The sum of the sides of a spherical polygon (not re-entrant) is less than 360°.* In other words: *The perimeter of any (non-re-entrant) spherical polygon is less than the length of a great circle.* This important proposition follows from Arts. 14 and 4. *b*.

(3) In an isosceles spherical triangle the angles opposite the equal sides are equal.

(4) The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the middle of the base is perpendicular to the base, and bisects the vertical angle.

(5) If two angles of a spherical triangle are unequal, the opposite sides are unequal, and the greater side is opposite the greater angle.

COR. If two edge angles of a triedral angle are unequal, the opposite face angles are unequal, and the greater face angle is opposite the greater diedral angle.

(6) If two sides of a spherical triangle are unequal, the opposite angles are unequal, and the greater angle is opposite the greater side.

Ex. Give the corresponding proposition for a triedral angle.

Propositions (3)–(6) can be proved in the same way as the corresponding propositions in plane geometry.

ON POLAR TRIANGLES.

16. ***a.*** NOTE. Three straight lines on a plane, no two of which are parallel, intersect in three points, and form one triangle. Three great circles on a sphere have six points of intersection, and form eight spherical triangles. Thus, on a globe, the equator and any two great circles through the poles have as intersections the two poles and the four points where the two great circles cross the equator; and there are eight triangles formed, namely, four in the northern hemisphere and four in the southern.

b. **Definitions.** If great circles be described with the vertices of a spherical triangle, say ABC (Fig. 14), as poles; and if there be taken that intersection of the circles described with B and C as poles which lies on the same side of BC as does A, namely A_1; and similarly for the other intersections; then a spherical triangle is formed, which is called **the polar triangle** of the first triangle ABC.

Two spherical polygons are *mutually equilateral* when the sides of the one are respectively equal to the sides of the other, whether taken in the same or in the reverse order; the polygons

are *mutually equiangular* when the angles of the one are respectively equal to the angles of the other, whether taken in the same or in the reverse order.

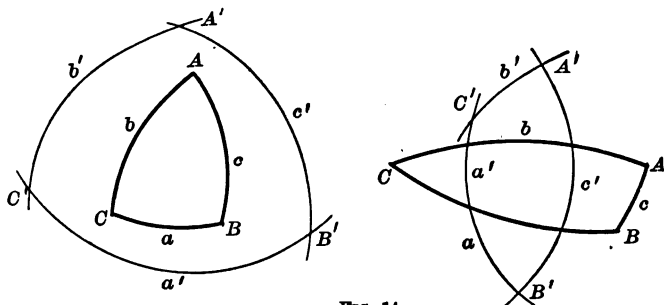



FIG. 14

c. **Proposition.** *If the first of two spherical triangles is the polar triangle of the second, then the second is the polar triangle of the first.*

If $A'B'C'$ (Fig. 14) is the polar triangle of ABC, then ABC is the polar triangle of $A'B'C'$. Since A is the pole of the arc $B'C'$, the point A is a quadrant's distance from B'. Also, since C is the pole of $B'A'$, the point C is a quadrant's distance from B'. Since B' is thus a quadrant's distance from both A and C, it is the pole of the arc AC (Art. 8). Similarly it can be shown that A' is the pole of the arc BC, and that C' is the pole of the arc AB. Hence ABC is the polar triangle of $A'B'C'$.

d. **Proposition.** *In two polar triangles, each angle of the one is the supplement of the side opposite to it in the other.*

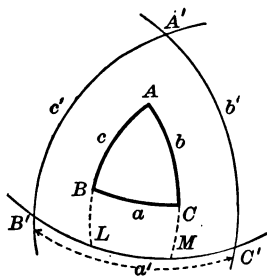



FIG. 15

Let ABC and $A'B'C'$ (Fig. 15) be a pair of polar triangles, in which A, B, C, A', B', C', are the angles, and a, b, c, a', b', c', are the sides. Then

$$A = 180 - a', \qquad A' = 180 - a,$$
$$B = 180 - b', \qquad B' = 180 - b,$$
$$C = 180 - c', \qquad C' = 180 - c$$

Produce the arcs AB and AC to meet $B'C'$ in L and M respectively.

Since B' is the pole of ACM, $B'M = 90°$; and

since C' is the pole of ABL, $LC' = 90°$.

Hence $B'M + LC' = 180°$;

that is, $B'M + MC' + LM = 180°$,

or $B'C' + LM = 180°$. (1)

Since A is the pole of the arc $B'C'$, the arc LM measures the angle A (Art. 11. *c*).

Hence, (1) becomes $A + a' = 180°$, or $A = 180° - a'$.

The other relations can be proved in a similar manner.

Cor. If two spherical triangles are mutually equiangular, their polar triangles are mutually equilateral. If two spherical triangles are mutually equilateral, their polar triangles are mutually equiangular.

Note. On account of the properties in (*d*), a triangle and its polar are sometimes called *supplemental triangles.*

e. **The use of the polar triangle.** Because of the fact that the sides and angles of a triangle are respectively supplementary to the angles and sides of its polar triangle, many relations can be easily derived by reference to the polar triangle. For, if a relation is true for spherical triangles in general, then it is true for the polar of any triangle. Let the relation be stated for the polar triangle; in this statement express the values of the sides and angles of the polar triangle in terms of the angles and sides of the original triangle; the statement thus derived expresses a new relation between the parts of the original triangle. This will be exemplified in later articles.

17. Proposition. *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*

Let ABC be any spherical triangle; it is required to show that

$$180° < A + B + C < 540°.$$

Construct the polar triangle $A'B'C'$. Then, by Art. 16. *d*,

$$A + a' = 180°, \ B + b' = 180°, \ C + c' = 180°.$$

Hence, on adding, $A + B + C + a' + b' + c' = 540°$,

or, $$A + B + C = 540° - (a' + b' + c').$$

Now [Art. 15 (2)] $a' + b' + c'$ is less than 360°, and greater than 0°.

$\therefore (A + B + C) = 540° -$ (something less than 360° and greater than 0°).

$$\therefore A + B + C > 540° - 360°, \text{ i.e. } A + B + C > 180°;$$

and $$A + B + C < 540° - 0°, \text{ i.e. } A + B + C < 540°.$$

18. Definitions. ***a.*** The amount by which the sum of the three angles of a spherical triangle is greater than 180° is called its **spherical excess.** It is shown in Art. 57 that the area of a triangle depends upon its spherical excess.

b. A spherical triangle may have two right angles, three right angles, two obtuse angles, or three obtuse angles. For example, on a globe the spherical triangle bounded by any arc (not 90°) on the equator and the arcs of the meridians joining the extremities of the former arc to the North Pole, has two right angles; if the arc on the equator is a quadrant, the triangle has three right angles. The polar of the triangle whose sides are 35°, 25°, 15°, has three obtuse angles. A spherical triangle having two right angles is called a *bi-rectangular triangle,* and a spherical triangle having three right angles is called a *tri-rectangular* triangle. A triangle having one side equal to a quadrant is called a *quadrantal* triangle; one having two sides each a quadrant is said to be *bi-quadrantal,* and one having each of its three sides equal to a quadrant is said to be *tri-quadrantal.*

c. **A lune** is a spherical surface bounded by the halves of two great circles. The *angle of the lune* is the angle made by the two great circles. For instance, on a globe the surface between the meridians 10° W. and 40° W. is a lune; the angle of this lune is equal to 30°. On the same circle or on equal circles lunes having equal angles are equal. (For they can evidently be made to coincide.)

19. On the convention that each side of a spherical triangle be less than 180°. In spherical geometry and trigonometry it is found convenient to restrict attention to triangles the sides of which are each less than a semicircle or 180°. (This convention can be set aside when it is necessary to consider what is called the *general spherical triangle*, in which an element may have any value from 0° to 360°.) A triangle such as $ADBC$ (Fig. 16) which has a side ADB greater than 180°, need not be considered; for its parts can be immediately deduced from the parts of ACB, each of whose sides is less than 180°. It is easily proved that if an angle of a spherical triangle is greater than 180°, the opposite side is also greater than 180°, and *vicê versâ*. Thus, in the triangle $ADBC$, if the angle ACB is greater than 180°, so is the side ADB; and if ADB is greater than 180°, so is the opposite angle. [*Suggestion for proof:* Produce the arc AC to meet the arc ADB.]

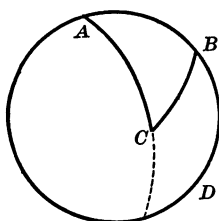



FIG. 16

20. Proposition. The shortest line *that can be drawn on the surface of a sphere between two given points is the arc of a great circle, not greater than a semicircle, which joins the points.*

Let A and B be any two points on a sphere, and let ACB be a great-circle arc not greater than a semicircle; then ACB is the shortest line that can be drawn from A to B on the sphere.

About A as a pole describe a circle DCE with radius AC, and about B as a pole describe a circle FCG with radius BC. It will be shown (1) that C is the only point which is common to both these circles; (2) that the shortest line that can be drawn from A to B on the surface must pass through C.

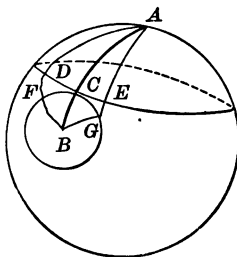
FIG. 17

(1) Take any point G, other than C, on the circle FCG. Draw the great-circle arcs AEG and BG. By Art. 15 (1),

$$AG + GB > AB;\ i.e.\ AG + GB > AC + CB.$$

Now $$AE = AC, \text{ and } GB = CB.$$

Hence $$AE + GB = AC + CB:$$

and, accordingly, $$AG > AE.$$

Therefore G is outside of the circle DCE. But G is any point (other than C) on the circle FCG. Hence C is the only point common to the circles DCE and FCG.

(2) Let $ADFB$ be any line drawn on the surface from A to B, but not passing through C. Whatever the character of the line AD may be, a line exactly like it can be drawn from A to C; and a line like BF can be drawn from B to C.

[This can be seen by regarding A-DCE as a cap fitting closely to the sphere, and supposing that this cap revolves about A until D is at C. Then a line exactly like AD is drawn from A to C.]

These lines being drawn, there will be a line from A to B which is less than $ADFB$ by the part DF. It has thus been proved that a line can be drawn from A to B through C which is shorter than any other line from A to B which does not pass through C. But C is *any* point on the great-circle arc from A to B. Hence the shortest line from A to B must pass through *every* point in ACB, and, accordingly, must be the arc ACB itself.

Note. This proposition can also be proved by the *method of limits*. It is shown that the length of any arc on a sphere is equal to the limit of the sum of the lengths of an infinite number of infinitesimal great-circle arcs inscribed in the given arc. (See Rouché et De Comberousse, *Traité de Géométrie*.) See Art. 6. *c*.

PROBLEMS OF CONSTRUCTION.

21. The actual making of the following constructions will add much to the clearness and vividness of the notions of most students about the surface of a sphere. An easy familiarity with the problems of Arts. 23, 24, which discuss the construction of triangles, will place the student in an advantageous position with respect to spherical trigonometry. This position is similar to that occupied by him, through his knowledge of the construction of plane triangles, when he entered upon the study of plane trigonometry. (See *Plane Trigonometry*, p. 20, Note, Art. 21, Art. 34 (to Case I.), Art. 53.)

N.B. The student should try to make these constructions for himself, and should fall back upon the book only as a last resort.

22. Problems on great circles.

(1) *To find the poles of a given great circle.* About any two points of the given circle as poles, describe great circles; their intersections will be the poles required (Art. 8).

(2) *To draw a great circle through two given points.* About the two given points as poles, describe great circles; about either of the intersections of these circles as a pole, describe a great circle; this will be the circle required. (See Arts. 8, 9.)

(3) *To cut from a great circle an arc $n°$ long.* Separate the points of the compasses by a distance equal to a chord which subtends a central angle of $n°$ in a circle whose radius is equal to the radius of the sphere; place the points of the compass on the great circle; the intercepted arc will be the one required.

(4) *To draw a great circle through a given point perpendicular to a given great circle.* Find a pole of the given circle by (1); draw a great circle through this pole and the given point by (2); this circle will be the one required (Art. 11. *d*).

(5) *To construct a great circle making a given angle with a given great circle, the point of intersection being given.* About the given point of intersection as pole, describe a great circle; on this circle lay off an arc, measured from the given circle, having as many (arcual) degrees as there are (angular) degrees in the given angle; draw a great circle through the extremity of this arc and the given point of intersection; this will be the circle required (Art. 11. *c*).

(6) *To construct a great circle passing through a given point and making a given angle with a given great circle.* [When the given point is on the given circle this problem reduces to problem (5).] It is easily shown that the angle between two great circles is equal to the angular distance (Art. 6. *c*) between their poles. Hence, find a pole of the given circle by (1); about this point as pole describe a second circle whose angular radius (Art. 6. *d*) is equal to the given angle; the pole of the required circle must be on this second circle. About the given point as pole describe a great circle; if the required problem is possible, this circle will either touch or intersect the second circle. The points of contact or intersection are the poles of two great circles, each of which will satisfy the given conditions.

Ex. Discuss the case in which the given point is the pole of the given circle.

23. Construction of triangles. The three sides and the three angles of a spherical triangle constitute its *six parts* or *elements*. *If any three of these six parts be known, the triangle can be constructed.* The *construction* belongs to geometry; the *computation* of the three remaining parts, when three parts are given, is an important part of spherical trigonometry. The sets of three parts that can be taken from the six parts of a spherical triangle are as follows:

I. **Three sides.**
II. **Three angles.**
III. **Two sides** and **their included angle.**
IV. **One side** and **the two adjacent angles.**
V. **Two sides** and **the angle opposite one of them.**
VI. **Two angles** and **the side opposite one of them.**

Note. There are *four* construction problems in the case of plane triangles (Plane Trig., Art. 53). When three angles of a spherical triangle are given, there is only one spherical triangle (with the triangle symmetrical to it), as will presently appear, which satisfies the given conditions. When three angles of a plane triangle are given, there is an infinite number of triangles, of the same shape, but of different magnitudes, which have angles equal to the three given angles. Cases IV. and VI. above, in which two angles are given, reduce to a single case in plane trigonometry, namely, the case in which one side and two angles are given; since the sum of the three angles of any plane triangle is 180°.

24. To construct a spherical triangle.

I. **Given the three sides.** On any great circle lay off an arc equal to one of the given sides [Art. 22 (3)]. About one extremity of this arc as pole, describe a circle with a radius (arcual) equal to the second of the given sides; about the other extremity of the arc as pole, describe a circle with a radius equal to the third of the given sides. By arcs of great circles join either of the points of intersection of the last two circles to the extremities of the arc first laid off; the triangle thus formed satisfies the given conditions.

Ex. 1. Compare the construction in the corresponding case in plane triangles.

Ex. 2. How many triangles are possible when the first arc is laid off? Are these triangles equal or symmetrical?

Ex. 3. Construct ABC: (a) Given $a = 70°$, $b = 65°$, $c = 40°$; (b) Given $a = 120°$, $b = 115°$, $c = 80°$.

Ex. 4. Determine approximately the angles of these triangles. (See Arts. 11. c, 34.)

II. **Given the three angles.** Calculate the sides of the polar triangle (Art. 16. *d*); construct it by I. above; construct *its* polar (Art. 16. *b*); the latter triangle is the one required.

Ex. 1. How many triangles can be drawn when one side of the polar triangle is fixed? Are these triangles equal or symmetrical?

Ex. 2. Discuss the corresponding case in plane triangles.

Ex. 3. Construct ABC: (a) Given $A = 85°$, $B = 75°$, $C = 55°$; (b) Given $A = 75°$, $B = 105°$, $C = 100°$.

Ex. 4. Determine approximately the sides of these triangles.

III. **Given two sides and their included angle.** Take any point on any great circle; through this point draw a circle making with the first circle an angle equal to the given included angle [Art. 22 (5)]; from the chosen point and on the first circle bounding this angle, lay off an arc equal to one of the given sides; from the same point and on the second circle bounding the angle, lay off an arc equal to the other given side. Join the extremities of these arcs by the arc of a great circle; the triangle thus formed is the one required.

Ex. 1. How many triangles can be made when the first circle and the point are chosen? Are these possible triangles equal or symmetrical?

Ex. 2. Discuss the corresponding case in plane triangles.

Ex. 3. Construct ABC: (a) Given $a = 75°$, $b = 120°$, $C = 65°$; (b) Given $b = 35°$, $c = 70°$, $A = 110°$.

Ex. 4. Determine approximately the remaining parts of these triangles.

IV. **Given a side and its two adjacent angles.**

Either: ***a.*** On any arc of a great circle lay off an arc equal to the given side; its extremities will be taken as two vertices of the required triangle. Through one extremity of this arc draw a great circle making with the arc an angle equal to one of the given angles; through the other extremity of the arc draw a

great circle making with the arc (and on the same side as the angle first constructed) an angle equal to the other of the given angles. The point of intersection of these two circles which is on the same side of the arc as the two angles, is the third vertex of the required triangle.

Or: **b.** Calculate the corresponding parts of the polar triangle; construct it by III.; construct *its* polar; this will be the triangle required.

Ex. 1. How many triangles are possible when the first arc is chosen? Are these triangles equal or symmetrical?

Ex. 2. Discuss the corresponding case in plane triangles.

Ex. 3. Solve Problem III. by means of IV. a, and the polar triangle.

Ex. 4. Construct ABC: (a) Given $a = 75°$, $B = 65°$, $C = 110°$; (b) Given $b = 110°$, $A = 40°$, $C = 63°$.

Ex. 5. Determine approximately the remaining parts of these triangles.

V. **Given two sides and the angle opposite to one of them.** [First, review the corresponding case in plane geometry.]

To construct a triangle ABC when a, b, A, are known: Through any point A of a great circle $ALA'A$ draw the semicircle, ACA', making an angle $A'LAC$ equal to the given angle A. From this semicircle cut off an arc AC equal to b. About C as a pole, with an arc equal (in degrees) to the side a, describe a circle. The intersection B of this circle with ALA' will be the third vertex of the required triangle, A and C being the other two vertices.

Four cases arise, viz.: —

(1) When the circle described about C fails to intersect ALA';

(2) When it just reaches to ALA';

(3) When it intersects the semicircle ALA' in but *one* point;

(4) When it intersects the semicircle ALA' in *two* points.

Case (1) is represented in Figs. 18, 22; case (2), in Figs. 19, 23; case (3), in Figs. 20, 24; and case (4), in Figs. 21, 25. In Figs. 18 and 22 the angle A is respectively acute and obtuse; and similarly for each of the other pairs of figures.

NOTE. In Figs. 18–25 AKA' is a great circle in the plane of the paper, and $ALA'A$ is a great circle at right angles to that plane, ALA' being above the surface of the paper, and the dotted AA' being below. In Figs. 18–21,

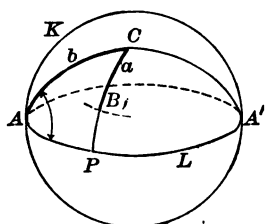



Fig. 18

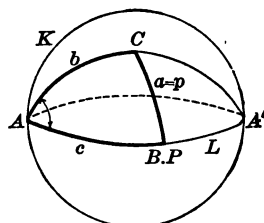
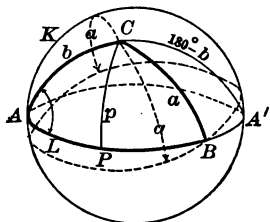



Fig. 19

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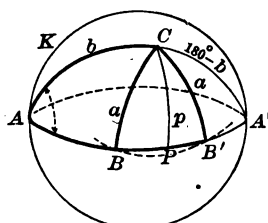
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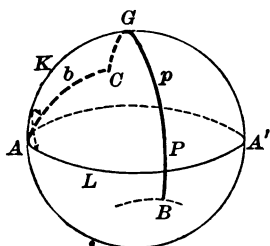
Fig. 22
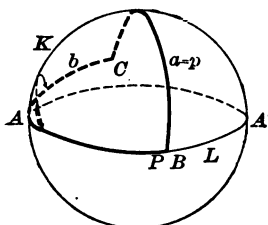
Fig. 23
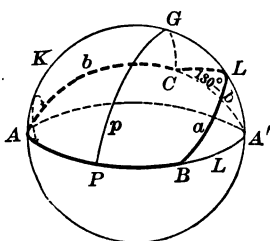
Fig. 24
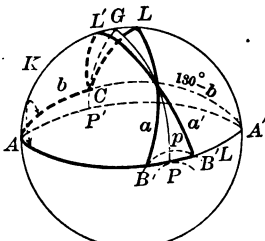
Fig. 25

angle A is acute [equal to $PAK(90°) - KAC$], and the arc AC is in front of the paper. In Figs. 22–25, angle A is obtuse [equal to $PAK(90°) + KAC$), and the arc AC is behind the paper.

In Fig. 21 there are two triangles (not equal or symmetrical) that satisfy the given conditions; and likewise in Fig. 25. Hence V. is **an ambiguous case** in spherical geometry.

In each figure let the perpendicular arc CP be drawn from C to the semicircle ALA', and let its length be called p. [See Ex. 1, p. 101.]

A. *When angle A is acute:*

Fig. 18 shows that the triangle required is impossible, if $CB < CP$, *i.e.* if $a < p$.

Fig. 19 shows that the triangle required is right angled if $CB = CP$, *i.e.* if $a = p$.

Fig. 20 shows that there is but one triangle which satisfies the given conditions, if

$$CB > CP,\ CB > CA, \text{ and } CB < CA';$$

$$i.e. \text{ if } a > p,\ a > b, \text{ and } a < 180° - b.$$

Similarly, there is only one triangle if $a > p$, $a < b$, and $a > 180° - b$.

Fig. 21 shows that there are two triangles which satisfy the given conditions, if

$$CB > CP,\ CB < CA, \text{ and } CB < CA';$$

$$i.e. \text{ if } a > p,\ a < b, \text{ and } a < 180° - b.$$

B. *When angle A is obtuse:*

Fig. 22 shows that the triangle required is impossible, if $CGB > CGP$, *i.e.* if $a > p$.

Fig. 23 shows that the triangle required is right angled, if $CGB = CGP$, *i.e.* if $a = p$.

Fig. 24 shows that there is but one triangle which satisfies the given conditions, if

$$CLB < CGP,\ CLB < CA \text{ and } CLB > CA';$$

$$i.e. \text{ if } a < p,\ a < b, \text{ and } a > 180° - b.$$

Similarly, there is only one triangle if $a < p$, $a > b$, and $a < 180° - b$;

Fig. 25 shows that there are two triangles which satisfy the given conditions, if

$$CLB < CGP,\ CLB > CA, \text{ and } CLB > CA';$$

$$i.e. \text{ if } a < p,\ a > b, \text{ and } a > 180° - b.$$

In Fig. 25 produce PGC to meet the great circle $ALA'A$ in P'. Then $CP' = 180° - p$. Since AC and CA' are each greater than CP', it follows that $a > 180° - p$.

It is also apparent from Figs. 20 and 21 that the triangle is impossible,

$$\text{if } A \text{ is acute, } a > b, \text{ and } a > 180^\circ - b;$$

and it is apparent from Figs. 24 and 25 that the triangle is impossible,

$$\text{if } A \text{ is obtuse, } a < b, \text{ and } a < 180^\circ - b.$$

Some special cases which may be investigated by the student, are indicated in the exercises on this chapter at page 101.

VI. **Given two angles and the side opposite one of them.** Calculate the corresponding parts of the polar triangle; construct it by V.; construct *its* polar; this is the required triangle. There may be two solutions, since the triangle first constructed may have two solutions.

Ex. 1. Construct ABC: (*a*) Given $a = 52°$, $b = 71°$, $A = 46°$; (*b*) Given $a = 99°$, $b = 64°$, $A = 95°$.

Ex. 2. Construct ABC: (*a*) Given $A = 46°$, $B = 36°$, $a = 42°$; (*b*) Given $A = 36°$, $B = 46°$, $a = 42°$.

Ex. 3. Determine approximately the remaining parts of these triangles.

N.B. *Questions and exercises on Chapter I. will be found at pages* 101–102.

CHAPTER II.

RIGHT-ANGLED SPHERICAL TRIANGLES.

25. Spherical trigonometry. Spherical trigonometry treats of the relations between the six parts of a triedral angle, or, what is the same thing (Art. 14), the relations between the six parts of the corresponding spherical triangle intercepted on the surface of the sphere. In Art. 24 it has been seen how a triangle can be *constructed* when any three parts are given; Chapters II. and III. are concerned with showing how the remaining parts can be *computed* when any three parts are known. In this chapter the relations between the sides and angles of a right-angled spherical triangle are deduced.* Throughout the book the relations between trigonometric ratios, discussed in plane trigonometry, will be employed.

26. Relations between the sides and angles of a right-angled spherical triangle.

Case I. *The sides about the right angle both less than* 90°.

Let ABC be a spherical triangle which is right angled at C and on a sphere whose centre is at O. First suppose that a and b are
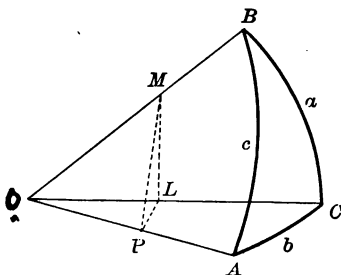
Fig. 26

* These relations can also be obtained from the relations, derived in Chapter III., between the parts of any spherical triangle or triedral angle.

each less than 90°. (It is easily shown, geometrically, that c is then less than 90°.) Draw OA, OB, OC. Take any point P on OA; in the plane OAC draw PL at right angles to OA, and let it meet OC in L; in the plane OAB draw PM at right angles to OA, and let it meet OB in M; and draw ML. Since PL and PM are perpendicular to the line OA, the line OA is perpendicular to the plane LPM (Euc. XI. 4); and, therefore, the plane LPM is perpendicular to the plane OAC (Euc. XI. 18). Also, the plane OCB is perpendicular to the plane OAC, since C is a right angle. Hence, LM, the intersection of the planes LPM and OCB, is perpendicular to the plane OAC (Euc. XI. 19); and hence, MLP and MLO are right angles.

In the triangle OPM, the angle OPM being right,

$$\cos POM = \frac{OP}{OM} = \frac{OP}{OL} \cdot \frac{OL}{OM}.$$

Now, $$\text{angle } POM = c, \quad \frac{OP}{OL} = \cos POL = \cos b,$$

and $$\frac{OL}{OM} = \cos LOM = \cos a.$$

$$\therefore \cos c = \cos a \cos b. \tag{1}$$

In the triangle LPM, angle $PLM = 90°$, and angle $LPM = A$;

$$\sin LPM = \frac{LM}{PM} = \frac{\frac{LM}{OM}}{\frac{PM}{OM}} = \frac{\sin LOM}{\sin POM}.$$

$$\therefore \sin A = \frac{\sin a}{\sin c}. \tag{2}$$

Also, $$\cos LPM = \frac{PL}{PM} = \frac{\frac{PL}{OP}}{\frac{PM}{OP}} = \frac{\tan POL}{\tan POM};$$

whence, $$\cos A = \frac{\tan b}{\tan c}. \tag{3}$$

Also, $$\tan LPM = \frac{LM}{PL} = \frac{\frac{LM}{OL}}{\frac{PL}{OL}} = \frac{\tan LOM}{\sin POL};$$

whence, $$\tan A = \frac{\tan a}{\sin b}. \qquad (4)$$

On remarking that A, a, denote an angle and its opposite side, and that b denotes the other side, the relations for angle B corresponding to (2), (3), (4), can be written immediately; viz.:

$$\sin B = \frac{\sin b}{\sin c} \quad (2'); \quad \cos B = \frac{\tan a}{\tan c} \quad (3'); \quad \tan B = \frac{\tan b}{\sin a} \quad (4').$$

These relations for B can also be deduced directly, by taking any point on OB and making a construction similar to that shown in Fig. 26.

Moreover,

$$\sin A = \tan A \cos A = \frac{\tan a}{\sin b} \cdot \frac{\tan b}{\tan c} = \frac{\tan a}{\tan c} \cdot \frac{1}{\cos b}. \quad \text{[By (3), (4).]}$$

$$\therefore \sin A = \frac{\cos B}{\cos b}. \quad \text{[By (3').]} \qquad (5)$$

Similarly, $$\sin B = \frac{\cos A}{\cos a}. \qquad (5')$$

Also, $$\cos c = \cos a \cos b = \frac{\cos A}{\sin B} \cdot \frac{\cos B}{\sin A}. \quad \text{[By (1), (5), (5').]}$$

$$\therefore \cos c = \cot A \cot B. \qquad (6)$$

For convenience of reference, relations (1)–(6) may be grouped together:

$$\mathbf{\cos c = \cos a \cos b.} \qquad (1)$$

$$\mathbf{\sin A = \frac{\sin a}{\sin c}.} \qquad \mathbf{\sin B = \frac{\sin b}{\sin c}.} \qquad (2), (2')$$

$$\mathbf{\cos A = \frac{\tan b}{\tan c}.} \qquad \mathbf{\cos B = \frac{\tan a}{\tan c}.} \qquad (3), (3')$$

$$\mathbf{\tan A = \frac{\tan a}{\sin b}.} \qquad \mathbf{\tan B = \frac{\tan b}{\sin a}.} \qquad (4), (4')$$

$$\mathbf{\sin A = \frac{\cos B}{\cos b}.} \qquad \mathbf{\sin B = \frac{\cos A}{\cos a}.} \qquad (5), (5')$$

$$\mathbf{\cos c = \cot A \cot B.} \qquad (6)$$

Case II. *The sides about the right angle both greater than* 90°.

In Fig. 27, C denotes the right angle, and the sides a, b, are each greater than a quadrant.
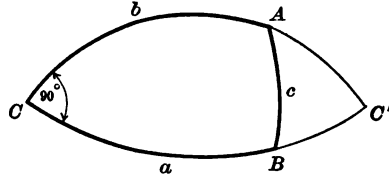
FIG. 27

Form the lune CC' by producing the sides a and b to meet in C'. Then ABC' is a right triangle in which the sides about the right angle are each less than 90°.

$$\therefore\ \cos c = \cos BC' \cos AC' = \cos (180 - a) \cos (180 - b).$$

Hence $$\cos c = \cos a \cos b.$$

Also,

$$\cos BAC' = \frac{\tan AC'}{\tan AB};\ i.e.\ \cos (180° - BAC) = \frac{\tan (180° - AC)}{\tan AB};$$

whence, $$\cos A = \frac{\tan b}{\tan c}.$$

In a similar manner the other relations in (1)–(6) can be shown to be true for ABC (Fig. 27).

NOTE. ABC' is said to be *co-lunar* with ABC. Every triangle has *three* co-lunar triangles, one corresponding to each angle.

Case III. *One of the sides about the right angle less than* 90°, *and the other side greater than* 90°.
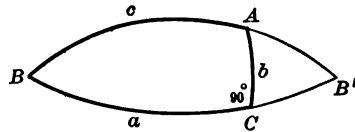
FIG. 28

In ACB let $C = 90°$, $a > 90°$, and $b < 90°$. Complete the lune BB'. Then ACB' is a right-angled triangle in which the sides about the right angle are each less than 90°.

In ACB', $\cos AB' = \cos AC \cos CB'$;

$$i.e.\ \cos(180^\circ - c) = \cos b \cos(180^\circ - a);$$

whence $\cos c = \cos a \cos b$.

Again,

$$\cos CAB' = \frac{\tan AC}{\tan AB'};\ i.e.\ \cos(180^\circ - BAC) = \frac{\tan AC}{\tan(180^\circ - BA)};$$

whence, $$\cos A = \frac{\tan b}{\tan c}.$$

In a similar way the other relations in (1)–(6) can be shown to be true for ABC (Fig. 28).

27. On species. Two parts of a spherical triangle are said to be *of the same species* (*or of the same affection*) when both are less than 90°, both greater than 90°, or both equal to 90°. Formula (1), Art. 26, shows that the *hypotenuse* of a right-angled spherical triangle is *less than* 90° when the *sides* about the right angle are *both greater or both less than* 90°; and it shows that the *hypotenuse is greater than* 90° when the *sides are of different species.* Formulas (4) and (4′) show that in a right-angled spherical triangle (since the sines of the sides are positive) *an angle and its opposite side are of the same species.* These important properties can also be deduced geometrically.

EXAMPLES.

N.B. It is advisable to remember the result of **Ex. 1.**

1. State relations (1)–(6), Art. 26, in words.

(1). *cos* hyp. = product of *cosines* of sides.

(6). *cos* hyp. = product of *cotangents* of angles.

(2), (2′). *sin* angle = *sin* opposite side ÷ *sin* hyp.

(3), (3′). *cos* angle = *tan* adjacent side ÷ *tan* hyp.

(4), (4′). *tan* angle = *tan* opposite side ÷ *sin* adjacent side.

(5), (5′). *cos* angle = *cos* opposite side × *sin* other angle.

[Compare (2), (3), (4), with the corresponding formulas in plane triangles.]

2. Deduce formulas (1)–(4) by means of a figure in which P is anywhere on OB (see Fig. 26).

3. Deduce formulas (1)–(4) by means of a figure in which P (see Fig. 26) is: (*a*) in OA produced; (*b*) in OB produced; (*c*) at the point A; (*d*) at the point B.

4. The two sides about the right angle of a spherical triangle are 60° and 75°; find the hypotenuse and the other angles by means of relations (1), (4), (4′), Art. 26. Check (or test) the result by means of other formulas.

5. In ABC, given $A = 47°30'$, $B = 53°40'$, $C = 90°$; find the remaining parts, and check the results.

6. Solve some of the examples in Art. 31, and check the results.

28. Solution of a right-angled triangle.

N.B. The student is advised to investigate this subject independently; and, *before* reading this article, to *put in writing* in an orderly manner his ideas about the solution of right triangles. These ideas will thus be made clearer in his mind, and his subsequent reading will be easier.

In a right triangle there are five elements beside the right angle. These five elements can be taken in groups of three in ten different ways. Each of these ten groups is involved in the ten relations derived in Art. 26; the three elements of each group are, accordingly, connected by one relation.

Ex. (*a*) Write all the groups of three that can be formed from a, b, c, A, B, such as a, b, c; a, b, A; etc.

(*b*) Write the relation connecting the elements of each group.

It follows that if any two elements of a right-angled spherical triangle besides the right angle be given, then the remaining three elements can be determined. The method of finding a third element is as follows:

Write the relation involving the two given elements and the required element; solve this equation for the required element.

Check (or *test*). When the required elements are obtained, the results can be checked by examining whether they satisfy relations which have not been employed in the solution, and, preferably, the relation involving the newly found elements.

E.g., suppose that A, b, are known, C being 90°; then a, c, B, are required. Side a can be found by (4); side c, by (3); and angle B, by (5). The values found for a, c, B, can be checked by (3′).

NOTE 1. The cosine, tangent, and co-tangent of sides and angles greater than 90°, are *negative*. *Careful attention must be paid to the algebraic signs of the trigonometric functions appearing in the work.*

NOTE 2. The properties stated in Art. 27 are very useful.

NOTE 3. Determine each element from the given elements alone. If an element is found erroneously and then used in finding a second element, the second element will also be wrong.

The ten possible groups of three elements correspond to the following *six* cases for solution, in which the given elements are respectively:

(1) Two sides.
(2) Hypotenuse and a side.
(3) Hypotenuse and an angle.
(4) Two angles.
(5) Side and opposite angle.
(6) Side and adjacent angle.

Before proceeding to the solution of numerical examples, it is necessary to refer more particularly to one of these cases; and also to call attention to the fact that the ten formulas for right triangles (Art. 26) may be grouped in two very simple and convenient rules.

29. The ambiguous case. When the given parts are a side and its opposite angle, there are two triangles which satisfy the given conditions. For, in ABC (Fig. 29), let $C = 90°$, and let A and CB (equal to a) be the given parts. Then, on completing the lune AA', it is evident that the triangle $A'BC$ also satisfies the given conditions, since $BCA' = 90°$, $A' = A$, and $CB = a$. The remaining parts in $A'BC$ are respectively supplementary to the remaining parts in ABC; thus $A'B = 180° - c$, $A'C = 180° - b$, $A'BC = 180° - ABC$.

FIG. 29

This ambiguity is also apparent from the relations (1)–(6), Art. 26. For, if a, A, are given, then the remaining parts, namely,

c, b, B, are all determined from their sines [see (2), (4), (5′),]; and, accordingly, c, b, B, may each be less or greater than 90°. On the other hand, if, for instance, a and c are given, then b is determined from its cosine by (1); and there is no ambiguity, because b is less or greater than 90° according as cos b is respectively positive or negative.

N.B. Both solutions should be given in the ambiguous case, unless some information is given which serves to indicate the particular solution that is suitable.

30. Napier's rules of circular parts. The ten relations derived in Art. 26 are all included in two statements, which are called *Napier's rules of circular parts*, after the man who first announced them, Napier, the inventor of logarithms.

Let ABC be a triangle right-angled at C. Either draw a right-angled triangle, and mark the sides and angles as in Fig. 31, or draw a circle and mark successive arcs as in Fig. 32, in which b, a, *Co-B*, *Co-c*, *Co-A*, are

FIG. 30 FIG. 31 FIG. 32

arranged in the same order as b, a, B, c, A, in Fig. 30. (Here *Co-B*, *Co-c*, *Co-A*, denote the complements of B, c, and A, respectively.) The five quantities, a, b, *Co-B*, *Co-c*, *Co-A*, are known as *circular parts*. That is, the right angle being omitted, the two sides and the complements of the hypotenuse and the other angles are called the circular parts of the triangle.

In Figs. 31 and 32 each part has two circular parts *adjacent* to it, and two circular parts *opposite* to it. Thus, on taking a, for instance, the adjacent parts are b, *Co-B*, and the opposite parts are *Co-c*, *Co-A*. If any three parts be taken, one of them is midway between the other two, and the latter are either its two adjacent parts or its two opposite parts. Thus, if a, b, *Co-A*, be taken, then b is the middle part, and a, *Co-A*, are the adjacent parts; if a, b, *Co-c*, be taken, then *Co-c* is the middle part, and a, b, are opposite parts.

Ex. Take each of the circular parts in turn, write its opposite parts and adjacent parts, thus getting *ten* sets in all.

Napier's rules of circular parts are as follows:

I. *The sine of the middle part is equal to the product of the tangents of the adjacent parts.*

II. *The sine of the middle part is equal to the product of the cosines of the opposite parts.*

(The *i*'s, *a*'s, and *o*'s are lettered thus, in order to aid the memory.)

These rules are easily *verified.* For example, on taking a for the middle part,

$$\sin a = \tan b \tan (90° - B) = \tan b \cot B. \quad \text{[See Art. 26 (4').]}$$

$$\sin a = \cos (90° - A) \cos (90° - c) = \sin A \sin c. \quad \text{[See Art. 26 (2).]}$$

Again, on taking *Co-A* for the middle part,

$$\sin (90° - A) = \tan b \tan (90° - c), \text{ i.e. } \cos A = \tan b \cot c. \quad \text{[See Art. 26 (3).]}$$

$$\sin (90° - A) = \cos a \cos (90° - B), \text{ i.e. } \cos A = \cos a \sin B. \quad \text{[See Art. 26 (5').]}$$

In a similar way each of the remaining three parts can be taken in turn for the middle part, and the remaining six relations of Art. 26 shown to agree with Napier's rules.*

NOTE. One may either memorize the relations in Art. 26 (or Ex. 1, Art. 27), or use Napier's rules. Opinions differ as to which is the better thing to do.

Ex. 1. Verify Napier's rules by showing that they include the 10 relations in Art. 26.

Ex. 2. Prove Napier's rules.

31. Numerical problems. In solving right triangles the procedure is as follows:

(1) Indicate the two given parts and the three required parts.

---

* This is only a *verification* of Napier's rules. One *proof* of the rules would consist of the derivation of the relations in Art. 26 *plus* this verification. These rules were first published by Napier in his work *Mirifici Logarithmorum Canonis Descriptio* in 1614. Napier indicated a geometrical method of proof, and deduced the rules as special applications of a more general proposition. They are something more than mere technical aids to the memory. For an explanation of this and of their wider geometrical interpretation, see Charles Hutton, *Course in Mathematics* (edited by T. S. Davies, London, 1843), Vol. II. pp. 24–26; Todhunter, *Spherical Trigonometry,* Art. 68; E. O. Lovett, *Note on Napier's Rules of Circular Parts* (Bulletin Amer. Math. Soc., 2d Series, Vol. IV. No. 10, July, 1898).

(2) Write the relations that will be employed in the solution, and note carefully the algebraic signs of the functions involved. The noting of these signs will serve to show (unless a part is determined from its sine) whether a required part is less or greater than 90°.

(3) For use as a *check*, write the relation involving the three required parts.

(4) Arrange the work as neatly and clearly as possible.

N.B. Attention may be directed to the notes in Art. 28. Also see *Plane Trigonometry*, Art. 27 (particularly p. 45, notes 2, 4–6), and Art. 59, p. 107.

NOTE. The trigonometric function of any angle can be expressed in terms of some trigonometric function of an angle less than 90°. See *Plane Trigonometry*, Art. 45.

EXAMPLES.

1. Solve the triangle ABC, given:

$$C = 90°, \quad a = 44° 30', \quad b = 71° 40'$$

Solution*: $c =$, $A =$, $B =$

Formulas:

$$\cos c = \cos a \cos b,$$
$$\tan A = \tan a \div \sin b,$$
$$\tan B = \tan b \div \sin a.$$

Check: $\cos c = \cot A \cot B$.

Logarithmic formulas: [If necessary; see *Pl. Trig.*, Art. 27, Note 6.]

$$\log \cos c = \log \cos a + \log \cos b,$$
$$\log \tan A = \log \tan a - \log \sin b,$$
$$\log \tan B = \log \tan b - \log \sin a,$$
$$\log \cos c = \log \cot A + \log \cot B \ (check).$$

$$\log \sin a = 9.84566 - 10$$
$$\log \cos a = 9.85324 - 10$$
$$\log \tan a = 9.99242 - 10$$
$$\log \sin b = 9.97738 - 10$$
$$\log \cos b = 9.49768 - 10$$
$$\log \tan b = 0.47969$$

$$\therefore \log \cos c = 9.35092 - 10$$
$$\log \tan A = 0.01504$$
$$\log \tan B = 0.63403$$
$$\therefore c = 77° \ 2'.1.$$
$$A = 45° 59'.5.$$
$$B = 76° 55'.5.$$

Check:
$$\therefore \log \cot A = 9.98497 - 10$$
$$\log \cot B = 9.36595 - 10$$
$$\therefore \log \cos c = 9.35092 - 10$$

---

* To be filled in.

Note. Spherical triangles, like plane triangles, can also be solved without the use of logarithms. (See *Plane Trigonometry*, examples in Arts. 27, 55–62.)

2. Solve the triangle ABC, given:

$C = 90°$, Solution: $c =$

$A = 57° 40'$, $b =$

$a = 48° 30'$. $B =$

Formulas: $\sin c = \dfrac{\sin a}{\sin A}$, *Check:* $\sin B = \dfrac{\sin b}{\sin c}$.

$\sin b = \dfrac{\tan a}{\tan A}$,

$\sin B = \dfrac{\cos A}{\cos a}$.

$\log \sin a = 9.87446 - 10$ $\therefore \log \sin c = 9.94763 - 10$

$\log \cos a = 9.82126 - 10$ $\log \sin b = 9.85459 - 10$

$\log \tan a = 0.05319$ $\log \sin B = 9.90697 - 10$

$\log \sin A = 9.92683 - 10$ $\therefore c = 62° 25'.4$, or $117° 34'.6$.

$\log \cos A = 9.72823 - 10$ $b = 45° 40'.9$, or $134° 19'.1$.

$\log \tan A = 0.19860$ $B = 53° 49'.3$, or $126° 10'.7$.

The check gives $\log \sin B = 9.90696$.

On combining the results according to the principles of Art. 27, the solutions are:

(1) $c = 62° 25'.4$, $b = 45° 40'.9$, $B = 53° 49'.3$;

(2) $c = 117° 34'.6$, $b = 134° 19'.1$, $B = 126° 10'.7$.

3. Solve Ex. 1 without using logarithms.

4. Given $c = 90°$, $A = 57° 40'$, $a = 108° 30'$. Show both by geometry and trigonometry why the solution is impossible.

5. Solve the triangle ABC, in which $C = 90°$, and *check the results*, given:

(1) $a = 36° 25' 30''$, $b = 85° 40'$; (2) $c = 120° 20' 30''$, $a = 47° 30' 40''$;

(3) $c = 78° 25'$, $A = 36° 42' 30''$; (4) $A = 63° 18'$, $B = 37° 47'$;

(5) $a = 76° 48'$, $A = 82° 36'$; (6) $b = 39° 50' 20''$, $A = 47° 50'$;

(7) $a = 47° 40'$, $A = 30° 43'$; (8) $b = 70° 30' 30''$, $B = 80° 40' 20''$;

(9) $a = 108° 45'$, $B = 37° 42'$; (10) $c = 78° 20'$, $B = 47° 50'$;

(11) $A = 110° 27'$, $B = 74° 36'$; (12) $a = 108° 42'$, $b = 63° 26'$;

(13) $A = 124° 30'$, $b = 25° 40'$; (14) $c = 84° 47'$, $b = 39° 43'$.

32. Solution of isosceles triangles and quadrantal triangles.

Isosceles Triangles. Plane isosceles triangles can be solved by means of right triangles, as shown in *Plane Trigonometry,* Art. 32. A spherical isosceles triangle can be solved in a similar way, namely, by dividing it into two right triangles by an arc drawn from the vertex at right angles to the base. This arc bisects the base and the vertical angle.

Quadrantal Triangles. The polar triangle of a quadrantal triangle (Art. 18) is right-angled (Art. 16. *d*). Hence, a quadrantal triangle may be solved by solving its polar triangle by Arts. 28, 31, and then computing the required parts of the quadrantal triangle by Art. 16. *d*.

EXAMPLES.

1. Solve the triangle ABC, in which A and B are equal, and *check the results*, given:

(1) $a = 54° 20'$, $c = 72° 54'$; (2) $a = 66° 29'$, $A = 50° 17'$;

(3) $a = 54° 30'$, $C = 71°$; (4) $c = 156° 40'$, $C = 187° 46'$.

2. Solve the triangle ABC, given:

(1) $c = 90°$, $C = 67° 12'$, $a = 123° 48' 4''$;

(2) $c = 90°$, $A = 136° 40'$, $B = 105° 47'$.

33.* Solution of oblique spherical triangles. It has been seen (*Plane Trigonometry,* Art. 34) that oblique plane triangles can be solved by means of right triangles. Oblique spherical triangles can also be solved by means of right spherical triangles. Relating to spherical triangles there are *six problems of computation;* these correspond to the six problems of construction discussed in Arts. 23, 24. If any three parts of a triangle are given, the triangle can be constructed and the remaining parts can be computed. The several cases for computation will now be solved with the help of right-angled triangles.†

(In the figures in this article the given parts are indicated by crosses.)

N.B. The student is advised to try to solve Cases II.–VI. *before* reading the text.

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* When time is limited this article may be omitted, or merely glanced over.

† Other methods of solving triangles are shown in Chap. IV.

Case I. Given the three sides. In ABC (Figs. 33, 34) let a, b, c, be given, and A, B, C, be required. From C draw CD at right angles to AB, or AB produced. Let the segments AD and DB be denoted by m and n, respectively. If the direction from A to B is taken as the positive direction along the arc AB, then m is positive in Fig. 33 and negative in Fig. 34, while n is positive in both figures.

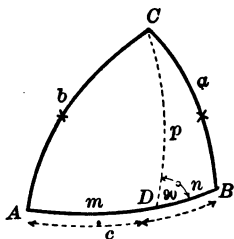
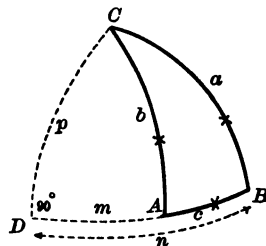



FIG. 33 FIG. 34

Special formula. In each figure,

$$\cos a = \cos n \cos p, \text{ and } \cos b = \cos m \cos p.$$

$$\therefore \cos p = \frac{\cos a}{\cos n} = \frac{\cos b}{\cos m}.$$

$$\therefore \frac{\cos n}{\cos m} = \frac{\cos a}{\cos b}.$$

$$\therefore \frac{\cos n - \cos m}{\cos n + \cos m} = \frac{\cos a - \cos b}{\cos a + \cos b}. \quad \text{[Composition and division.]}$$

From this, on applying *Plane Trigonometry*, Art. 52, formulas (7), (8),

$$\tan \tfrac{1}{2}(n + m) \tan \tfrac{1}{2}(n - m) = \tan \tfrac{1}{2}(a + b) \tan \tfrac{1}{2}(a - b). \quad (1)$$

Now $n + m = c$; hence, $n - m$ can be found by (1). Then the segments m and n can each be determined. The triangles ADC and BDC can then be solved by Arts. 28, 31; and the solution of ABC can be obtained therefrom.

Ex. 1. Solve Exs. 1, 2, Art. 42, by the method outlined above.

Ex. 2. Show how to solve this case when the perpendicular is drawn from A to BC.

Case II. Given the three angles. Solve the polar triangle by the method used in Case I.; and therefrom (Art. 16. *d*) compute the parts of the original triangle.

Ex. Solve Exs. 1, 2, Art. 43, by this method.

Case III. Given two sides and their included angle. In ABC (Fig. 35) let a, c, B, be given. Draw AD at right angles to BC, or BC produced.

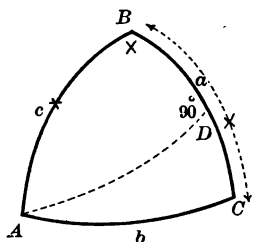



FIG. 35

In ABD, c and B are known; hence, BAD, AD, and BD can be found. In ADC, AD and DC (equal to $a - BD$) are now known; hence DAC, ACD, and AC can be found. Also, $CAB = CAD + DAB$. The student can examine the case in which AD falls outside ABC.

Ex. 1. Show how to solve the triangle by drawing a perpendicular arc from C to AB.

Ex. 2. Solve Exs. 1, 2, Art. 44, by means of right triangles.

Case IV. Given a side and the two adjacent angles. Two methods of solution may be employed.

Either: (1) Solve the polar triangle by the method used in Case III.; and therefrom compute the parts of the original triangle.

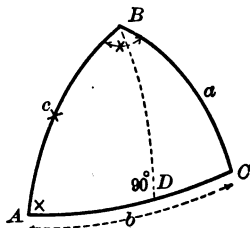
Or: (2) In ABC (Fig. 36) let A, B, c, be given. Draw the arc BD at right angles to AC. In ADB, AD, DB, and ABD can be found, since A and AB are known. Now $DBC = ABC - ABD$. In DBC, DB and DBC are now known; hence BC, CD, and C can be found. Then $AC = AD + DC$.

The student can examine the case in which BD falls outside ABC.

FIG. 36

Ex. 1. Solve the triangle by drawing a different perpendicular.

Ex. 2. How may solution (2) aid in the solution of Case III.?

Ex. 3. Solve Exs. 1, 2, Art. 45, by means of right triangles.

Case V. Given two sides and the angle opposite one of them. (*This may be an ambiguous case;* see Art. 24, V.)

In ABC (Fig. 37) let a, c, A, be given. From B draw the arc BD at right angles to AC to meet AC or AC produced. In ABD, c and A are known; hence AD, DB, and ABD can be found. In DBC, DB and a are now known; hence DBC, C, and DC can be found. Then $AC = AD + DC$, and $ABC = ABD + DBC$.

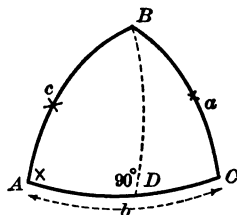



FIG. 37

Ex. 1. Examine the cases in which BD falls outside ABC.

Ex. 2. Examine the case in which *two* triangles satisfy the given conditions.

Ex. 3. Solve Exs. 1, 2, Art. 46, by means of right triangles.

Case VI. Given two angles and the side opposite one of them. Like Case V. this may be ambiguous; see Art. 24, VI. Two methods of solution may be employed.

Either: (1) Solve the polar triangle by the method used in Case V.; and therefrom compute the parts of the original triangle.

Or: (2) In ABC (Fig. 38) let A, C, c, be known. From B draw BD at right angles to AC to meet AC, or AC produced. Solve the triangle ABD; then solve the triangle DBC. The parts of ABC can be computed from these solutions.

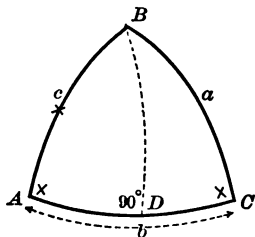



FIG. 38

Ex. 1. How may (2) aid in the solution of Case V.?

Ex. 2. Solve Exs. 1, 2, Art. 47, by means of right triangles.

Ex. 3. Solve the numerical examples in Art. 24.

34. Graphical solution of (oblique and right) spherical triangles. A plane triangle can be solved graphically by drawing to scale a triangle that satisfies the given conditions, and then measuring the required parts directly from the figure (*Plane Trigonometry,*

Arts. 10, 21). A spherical triangle can be solved graphically by drawing (Art. 24) upon any sphere a triangle that satisfies the given conditions, and then measuring the required parts of the triangle. The sides and angles (see Art. 11. *c*) can be measured with a thin, flexible, brass ruler, on which a length equal to a quadrant or a semicircle of the sphere is graduated from 0° to 90° or 180° respectively.

Small slated globes can be obtained fitting into hemispherical cups, whose rims are graduated from 0° to 180° in both directions. With such a globe, cup, and a pair of compasses, the constructions discussed in Art. 24 and the measurements referred to in this article are easily made.

If the student has the means at hand, it is advisable for him to solve some of the numerical problems graphically.

N.B. *Questions and exercises on Chapter II. will be found at p.* 102.

CHAPTER III.

RELATIONS BETWEEN THE SIDES AND ANGLES OF SPHERICAL TRIANGLES.

35. In this chapter some relations between the sides and angles of any spherical triangle (whether right-angled or oblique) will be derived. In the next chapter these relations will be used in the solution of practical numerical problems. The first two general relations (namely, the *Law of Sines* and the *Law of Cosines*), which are by far the most important, can be derived in various ways. In a short course it may be best to deduce these laws by means of the properties of right-angled triangles as set forth in Art. 26; and, accordingly, this method is adopted here. These laws are also derived directly from geometry in Note A at the end of the book. It may be stated here that the geometrical derivation will strengthen the student's understanding of the subject, and will show more clearly the correspondence (Art. 14) between the parts of a spherical triangle and the parts of a triedral angle.

36. The *Law of Sines* and the *Law of Cosines* deduced by means of the relations of right-angled triangles.

***A.* Derivation of the Law of Sines.**

Let ABC (Figs. 39, 40) be any spherical triangle. From B draw the arc BD at right angles to AC to meet AC, or AC produced, in D.

In ABD, $\quad \sin BD = \sin c \sin A$;

in CBD, $\quad \sin BD = \sin a \sin C$ (Fig. 39)

$= \sin a \sin BCD$ (Fig. 40) $= \sin a \sin BCA$.

Hence, in both figures, $\sin a \sin C = \sin c \sin A$.

$$\therefore \frac{\sin a}{\sin A} = \frac{\sin c}{\sin C}.$$

Similarly, by drawing an arc from C at right angles to AB, it can be shown that $\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}$.

$$\therefore \frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}. \qquad (1)$$

In words: *In a spherical triangle the sines of the sides are proportional to the sines of the opposite angles.* (Compare *Plane Trigonometry*, Art. 54, I.)

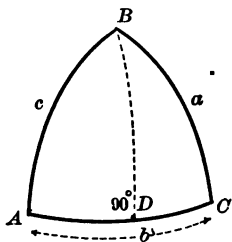



FIG. 39

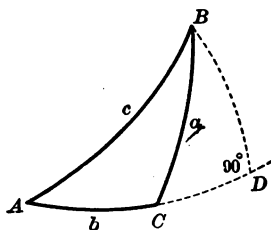



FIG. 40

***B.* Derivation of the Law of Cosines.**

$$\begin{aligned}\cos BC &= \cos CD \cos DB\\ &= \cos(b - AD)\cos DB, \text{ or } \cos(AD - b)\cos DB\\ &= \cos b \cos AD \cos DB + \sin b \sin AD \cos DB. \qquad (a)\end{aligned}$$

But $$\cos AD \cos DB = \cos c;$$

and $$\sin AD \cos DB = \frac{\cos c \sin AD}{\cos AD} = \cos c \tan AD$$

$$= \cos c \tan c \cos A = \sin c \cos A.$$

Hence, on substituting in (a),

$$\mathbf{\cos a = \cos b \cos c + \sin b \sin c \cos A.} \qquad (2)$$

Similarly, or by taking the sides in turn,

$$\cos b = \cos c \cos a + \sin c \sin a \cos B,$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

In words: *In a spherical triangle the cosine of any side is equal to the product of the cosines of the other two sides plus the product of the sines of these two sides and the cosine of their included angle.* (Compare *Plane Trigonometry*, Art. 54, II.)

NOTE 1. **The law of cosines, (2), is the fundamental and the most important relation in spherical trigonometry.** For, as shown in Note A, it can be deduced directly; the law of sines, (1), can be deduced from it; all other relations follow from these; and the relations for right triangles, Art. 26, can be deduced from the relations for triangles in general, on letting C be a right angle. The formulas for $\cos a$, $\cos b$, $\cos c$, were known to the Arabian astronomer Al Battani in the ninth century. (See *Plane Trigonometry*, p. 166.)

***C*. The Law of Cosines for the angles.** Relation (2) holds for all triangles, and, accordingly, for $A'B'C'$, the polar triangle of ABC. (See Fig. 14, Art. 16.) That is,

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$

$$\therefore \cos(180° - A) = \cos(180° - B)\cos(180° - C)$$
$$+ \sin(180° - B)\sin(180° - C)\cos(180° - a). \quad \text{[Art. 16. } d.]$$

$$\therefore -\cos A = (-\cos B)(-\cos C) + \sin B \sin C(-\cos a).$$

$$\therefore \mathbf{\cos A = -\cos B \cos C + \sin B \sin C \cos a.} \quad (3)$$

Similarly, $\cos B = -\cos C \cos A + \sin C \sin A \cos b,$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

Relation (3) can also be derived by means of right-angled triangles.

NOTE 2. From (2), $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}.$

The denominator in the second member is always positive. *If a differs more from 90° than does b*, then $\cos a$ is numerically greater than $\cos b$, and, accordingly, greater than $\cos b \cos c$; hence $\cos A$ and $\cos a$ have the same sign, and thus, *a and A are in the same quadrant.*

Similarly, a and A are in the same quadrant when a differs more from 90° than does c.

From (3), in which $\cos a = \dfrac{\cos A + \cos B \cos C}{\sin B \sin C},$

it can be shown in a similar way that *if A differs more from 90° than does B or C, then a and A are in the same quadrant.*

Ex. 1. Derive $\cos b$ and $\cos c$ by means of right triangles.

Ex. 2. Derive $\cos A$ and $\cos B$ by means of right triangles.

Ex. 3. Derive $\cos C$ from $\cos c$ by means of the polar triangle.

37. Formulas for the half-angles and the half-sides.

[Compare the method and results of this article with those of Art. 62, *Plane Trigonometry*].

I. The half-angles.

From Art. 36, (2), $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$. (1)

$$\therefore\ 1 - \cos A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c}$$

$$= \frac{\cos (b - c) - \cos a}{\sin b \sin c}.$$

$$\therefore\ 2 \sin^2 \tfrac{1}{2} A = \frac{2 \sin \frac{1}{2}(a - b + c) \sin \frac{1}{2}(a + b - c)}{\sin b \sin c}.$$

[*Plane Trigonometry*, Art. 52, (8).]

On putting $a + b + c = 2s$, then $-a + b + c = 2(s - a)$,

$$a - b + c = 2(s - b), \text{ and } a + b - c = 2(s - c).$$

$$\therefore\ \sin^2 \tfrac{1}{2} A = \frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}. \quad (2)$$

Similarly, from (1),

$$1 + \cos A = 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos a + \sin b \sin c - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos a - \cos (b + c)}{\sin b \sin c}.$$

$$\therefore\ 2 \cos^2 \tfrac{1}{2} A = \frac{2 \sin \frac{1}{2}(a + b + c) \sin \frac{1}{2}(b + c - a)}{\sin b \sin c}.$$

$$\therefore\ \cos^2 \tfrac{1}{2} A = \frac{\sin s \sin (s - a)}{\sin b \sin c}. \quad (3)$$

$$\left.\begin{aligned}\sin\tfrac{1}{2}A &= \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}};\\ \cos\tfrac{1}{2}A &= \sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}};\\ \text{and hence,}\quad \tan\tfrac{1}{2}A &= \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}.\end{aligned}\right\}\quad(4)$$

Therefore, $\tan\frac{1}{2}A = \dfrac{1}{\sin(s-a)}\sqrt{\dfrac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}};$

$$\left.\begin{aligned}\text{hence, if}\quad \tan r &= \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},\\ \text{then}\quad \tan\tfrac{1}{2}A &= \frac{\tan r}{\sin(s-a)}.\end{aligned}\right\}\quad(5)$$

Like formulas can be similarly derived for $\frac{1}{2}B$ and $\frac{1}{2}C$; or they may be written immediately on observing the symmetry in formulas (4) and (5); namely,

$$\left.\begin{aligned}\sin\tfrac{1}{2}B &= \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin a\sin c}}, & \sin\tfrac{1}{2}C &= \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}},\\ \cos\tfrac{1}{2}B &= \sqrt{\frac{\sin s\sin(s-b)}{\sin a\sin c}}, & \cos\tfrac{1}{2}C &= \sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}},\\ \tan\tfrac{1}{2}B &= \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s\sin(s-b)}}, & \tan\tfrac{1}{2}C &= \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}}.\end{aligned}\right\}\quad(6)$$

$$\tan\tfrac{1}{2}B = \frac{\tan r}{\sin(s-b)},\qquad \tan\tfrac{1}{2}C = \frac{\tan r}{\sin(s-c)}.\qquad(7)$$

It is shown in Art. 50 that r is the radius of the circle inscribed in the triangle ABC. Article 50 may be read at this time.

NOTE. By geometry, $2s < 360°$ and $b + c > a$. Hence, $s - a$ is positive and less than 180°. Similarly, $s - b$, $s - c$, are positive. Therefore, the quantities under the radical signs are positive. The positive signs must be given to each radical, for A, B, C, are each less than 180°, and, consequently, $\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$, are each between 0° and 90°.

EXAMPLES.

1. Derive each of the above formulas.
2. Given $a = 58°$, $b = 80°$, $c = 96°$. Find A, B, C.
3. Given $a = 46° 30'$, $b = 62° 40'$, $c = 83° 20'$. Find A, B, C.

The results in Exs. 2, 3, may be *checked* by Art. 36, (1).

II. The half-sides. From Art. 36, (3),

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

On finding $1 - \cos a$ and $1 + \cos a$, combining and simplifying in the manner followed for the half-angles, and putting

$$A + B + C = 2S,$$

the following formulas are obtained:

$$\left.\begin{aligned} \sin \tfrac{1}{2} a &= \sqrt{\frac{-\cos S \cos (S - A)}{\sin B \sin C}}; \\ \cos \tfrac{1}{2} a &= \sqrt{\frac{\cos (S - B) \cos (S - C)}{\sin B \sin C}}; \\ \tan \tfrac{1}{2} a &= \sqrt{\frac{-\cos S \cos (S - A)}{\cos (S - B) \cos (S - C)}}. \end{aligned}\right\} \quad (8)$$

Let
$$\left.\begin{aligned} \tan R &= \sqrt{\frac{-\cos S}{\cos (S - A) \cos (S - B) \cos (S - C)}}, \\ \text{then} \quad \tan \tfrac{1}{2} a &= \tan R \cos (S - A). \end{aligned}\right\} \quad (9)$$

Similarly, or from (8) and (9) by symmetry,

$$\left.\begin{aligned} \sin \tfrac{1}{2} b &= \sqrt{\frac{-\cos S \cos (S - B)}{\sin A \sin C}}, & \sin \tfrac{1}{2} c &= \sqrt{\frac{-\cos S \cos (S - C)}{\sin A \sin B}}. \\ \cos \tfrac{1}{2} b &= \sqrt{\frac{\cos (S - A) \cos (S - C)}{\sin A \sin C}}, & \cos \tfrac{1}{2} c &= \sqrt{\frac{\cos (S - A) \cos (S - B)}{\sin A \sin B}}, \\ \tan \tfrac{1}{2} b &= \sqrt{\frac{-\cos S \cos (S - B)}{\cos (S - A) \cos (S - C)}}, & \tan \tfrac{1}{2} c &= \sqrt{\frac{-\cos S \cos (S - C)}{\cos (S - A) \cos (S - B)}}. \end{aligned}\right\} \quad (10)$$

$$\tan \tfrac{1}{2} b = \tan R \cos (S - B), \qquad \tan \tfrac{1}{2} c = \tan R \cos (S - C). \qquad (11)$$

It is shown in Art. 49 that R is the radius of the circumscribing circle of the triangle ABC. Article 49 may be read at this time.

NOTE 1. Formulas (8)–(11) can also be derived from formulas (4)–(7) by making use of the polar triangle, as done in Art. 36. *C*.

NOTE 2. Since $A + B + C$ lies between 180° and 540° (Art. 17), S lies between 90° and 270°; hence, $\cos S$ is negative, and, accordingly, $-\cos S$ is positive. Since all the other functions under the radical signs are positive, the whole expression under each radical sign is positive.

NOTE 3. The positive value of the radical is taken, since each side (Art. 19) is less than 180°

EXAMPLES.

1. Derive formulas (10) from the values of $\cos b$ and $\cos c$.

2. Derive formulas (10) from formulas (6) by means of the polar triangle.

3. In ABC, given $A = 78° 40'$, $B = 63° 50'$, $C = 46° 20'$. Find a, b, c.

[SUGGESTION. Either use formulas (8)–(10); or, solve the polar triangle, and thence obtain the parts of the original triangle. [*The results may be checked by using both these methods*, or by Art. 36, (1).]

4. In ABC, given $A = 121°$, $B = 102°$, $C = 68°$. Find a, b, c.

5. Show that $\cos(S - A)$ is positive.

38. Napier's Analogies. On *dividing* $\tan \frac{1}{2} A$ by $\tan \frac{1}{2} B$ (Art. 37), there is obtained,

$$\frac{\tan \frac{1}{2} A}{\tan \frac{1}{2} B} = \frac{\sin(s-b)}{\sin(s-a)}.$$

From this, by composition and division,

$$\frac{\tan \frac{1}{2} A + \tan \frac{1}{2} B}{\tan \frac{1}{2} A - \tan \frac{1}{2} B} = \frac{\sin(s-b) + \sin(s-a)}{\sin(s-b) - \sin(s-a)}.$$

This, by *Plane Trigonometry*, Arts. 44. *B*, 52 (also, see Art. 61), reduces to

$$\frac{\sin \frac{1}{2} A \cos \frac{1}{2} B + \cos \frac{1}{2} A \sin \frac{1}{2} B}{\sin \frac{1}{2} A \cos \frac{1}{2} B - \cos \frac{1}{2} A \sin \frac{1}{2} B} = \frac{2 \sin \frac{1}{2}(2s - a - b) \cos \frac{1}{2}(a-b)}{2 \cos \frac{1}{2}(2s - a - b) \sin \frac{1}{2}(a-b)};$$

and thence, to

$$\frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2}(a-b)}. \qquad (1)$$

On *multiplying* $\tan \frac{1}{2} A$ by $\tan \frac{1}{2} B$, there is obtained

$$\tan \tfrac{1}{2} A \tan \tfrac{1}{2} B = \frac{\sin(s-c)}{\sin s},$$

i.e.

$$\frac{\sin \frac{1}{2} A \sin \frac{1}{2} B}{\cos \frac{1}{2} A \cos \frac{1}{2} B} = \frac{\sin(s-c)}{\sin s}.$$

From this, by composition and division,

$$\frac{\cos \frac{1}{2} A \cos \frac{1}{2} B - \sin \frac{1}{2} A \sin \frac{1}{2} B}{\cos \frac{1}{2} A \cos \frac{1}{2} B + \sin \frac{1}{2} A \sin \frac{1}{2} B} = \frac{\sin s - \sin(s-c)}{\sin s + \sin(s-c)}$$

$$= \frac{2 \cos \frac{1}{2}(2s - c) \sin \frac{1}{2} c}{2 \sin \frac{1}{2}(2s - c) \cos \frac{1}{2} c}.$$

Whence,

$$\frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2}(a+b)}. \qquad (2)$$

Either, on proceeding in a similar way with $\tan\frac{1}{2}a$ and $\tan\frac{1}{2}b$ [Art. 37, (8), (10)], *or*, on applying (1) and (2) to the polar triangle, there is obtained,

$$\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A-B)} \qquad (3)$$

and

$$\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A+B)}. \qquad (4)$$

Relations (1)–(4) are known as *Napier's Analogies.**

NOTE 1. Compare (3) with formula (2) Art. 61, *Plane Trigonometry.*

NOTE 2. The numerators in (3) are always positive, since $a+b+c<360°$ and $C<180°$. It follows, accordingly, that *$a-b$ and $A-B$ must have the same sign.* This also follows from the geometrical fact [Art. 15, (5)] that the greater angle is opposite the greater side.

NOTE 3. In relation (4), $\cot\frac{1}{2}C$ and $\cos\frac{1}{2}(a-b)$ are positive quantities; hence $\cos\frac{1}{2}(a+b)$ and $\tan\frac{1}{2}(A+B)$ have the same sign; and, accordingly, *$\frac{1}{2}(a+b)$ and $\frac{1}{2}(A+B)$ are of the same species* (Art. 27).

NOTE 4. *Derivation of* (3) *by applying* (1) *to the polar triangle.* On applying (1) to the polar triangle $A'B'C'$ (Fig. 14, Art. 16),

$$\frac{\sin\frac{1}{2}(A'+B')}{\sin\frac{1}{2}(A'-B')}=\frac{\tan\frac{1}{2}c'}{\tan\frac{1}{2}(a'-b')}.$$

$$\therefore \frac{\sin\frac{1}{2}(\overline{180°-a}+\overline{180°-b})}{\sin\frac{1}{2}(\overline{180°-a}-\overline{180°-b})}=\frac{\tan\frac{1}{2}(180°-C)}{\tan\frac{1}{2}(\overline{180°-A}-\overline{180°-B})}; \text{ [Art. 16. } d.]$$

i.e.

$$\frac{\sin(180°-\frac{1}{2}\overline{a+b})}{\sin\frac{1}{2}(b-a)}=\frac{\tan(90°-\frac{1}{2}C)}{\tan\frac{1}{2}(B-A)}.$$

Whence,

$$\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A-B)}.$$

NOTE 5. For a *geometrical deduction* of Napier's Analogies and the formulas in Art. 39, see M'Clelland and Preston, *Treatise on Spherical Trigonometry*, Part I., Art. 56, and the article *Trigonometry* in the Encyclopedia Britannica (9th edition).

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* That is, Napier's *proportions.* For a long time the word *analogy* was used in English in one of its original Greek meanings, namely, a proportion (*i.e.* an equality of ratios). This use of the word is now obsolete, and is only retained in a few phrases such as the above. Napier (see Art. 30, and *Plane Trigonometry*, Art. 1) discovered these proportions and gave them in his work, *Mirifici logarithmorum canonis descriptio*, in 1614

EXAMPLES.

1. Express Napier's Analogies in *words*.
2. *Write* the analogies involving B and C, A and C, b and c, a and c.
3. *Derive* some of the analogies in Ex. 2.

39. Delambre's Analogies or Gauss's Formulas.

By *Plane Trigonometry*, Art. 46, (1),

$$\sin \tfrac{1}{2}(A+B) = \sin \tfrac{1}{2} A \cos \tfrac{1}{2} B + \cos \tfrac{1}{2} A \sin \tfrac{1}{2} B.$$

By Art. 37, (4), (6),

$$\sin \tfrac{1}{2} A \cos \tfrac{1}{2} B = \frac{\sin (s-b)}{\sin c}\sqrt{\frac{\sin s \sin (s-c)}{\sin a \sin b}} = \frac{\sin (s-b)}{\sin c} \cos \tfrac{1}{2} C,$$

and
$$\cos \tfrac{1}{2} A \sin \tfrac{1}{2} B = \frac{\sin (s-a)}{\sin c}\sqrt{\frac{\sin s \sin (s-c)}{\sin a \sin b}} = \frac{\sin (s-a)}{\sin c} \cos \tfrac{1}{2} C.$$

$$\therefore \sin \tfrac{1}{2}(A+B) = \frac{\sin (s-a) + \sin (s-b)}{\sin c} \cos \tfrac{1}{2} C$$

$$= \frac{2 \sin \tfrac{1}{2}(2s-a-b) \cos \tfrac{1}{2}(a-b)}{2 \sin \tfrac{1}{2} c \cos \tfrac{1}{2} c} \cos \tfrac{1}{2} C.$$

$$\therefore \sin \tfrac{1}{2}(A+B) = \frac{\cos \tfrac{1}{2}(a-b)}{\cos \tfrac{1}{2} c} \cos \tfrac{1}{2} C. \qquad (1)$$

In a similar way it may be shown that

$$\sin \tfrac{1}{2}(A-B) = \frac{\sin \tfrac{1}{2}(a-b)}{\sin \tfrac{1}{2} c} \cos \tfrac{1}{2} C, \qquad (2)$$

$$\cos \tfrac{1}{2}(A+B) = \frac{\cos \tfrac{1}{2}(a+b)}{\cos \tfrac{1}{2} c} \sin \tfrac{1}{2} C, \qquad (3)$$

$$\cos \tfrac{1}{2}(A-B) = \frac{\sin \tfrac{1}{2}(a+b)}{\sin \tfrac{1}{2} c} \sin \tfrac{1}{2} C. \qquad (4)$$

Formulas (1)–(4) are known as Delambre's Analogies, and also as Gauss's Formulas or Equations.*

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* These formulas were discovered by Karl Friedrich Gauss (1777–1855), one of the greatest of German mathematicians and astronomers, and published without proof in his *Theoria Motus Corporum Cœlestium* in 1809; thus they bear his name. They were, however, published earlier by Karl Brandon Mollweide of Leipzig (1774–1825) in Zach's *Monatliche Correspondenz* for November, 1808. They were earliest discovered by Jean Baptiste Joseph Delambre (1749–1822), a great French astronomer, in 1807, and published in the *Connaissance des Temps* in 1808. The geometrical proof (see Note 5, Art. 38) was the one originally given by Delambre. This proof was rediscovered and announced by M. W. Crofton in 1869, and published in the *Proceedings of the London Math. Soc.*, Vol. III. (1869–1871), p. 13.

NOTE 1. Equations (3) and (4) can also be derived by applying (1) and (2) to the polar triangle.

NOTE 2. Delambre's Analogies can also be deduced by help of Napier's Analogies. (See Todhunter, *Spherical Trigonometry*, Art. 54; *Nature*, Vol. XL. (1889, Oct. 31), p. 644.)

NOTE 3. On the other hand, Napier's Analogies can be easily derived from Delambre's Analogies; namely, on dividing corresponding members, one by the other, in (1) and (3), (2) and (4), (4) and (3), (2) and (1).

EXAMPLES.

1. Write Delambre's Analogies involving B and C, and C and A.
2. Derive (3) and (4) from (1) and (2), using the polar triangle.
3. Derive Delambre's Analogies from Napier's Analogies.
4. Derive some of the analogies in Ex. 1 directly.

40. Other relations between the parts of a spherical triangle. The preceding articles of this chapter present few more relations than are required for the solution of spherical triangles. Between the parts of a spherical triangle there are many other relations which are interesting and useful for many purposes, and which either set forth, or lead to the discovery of, important geometrical properties * of spherical triangles.

For example, if in equation (2) Art. 36, the value of $\cos c$ in the second equation that follows, be substituted, then

$$\cos a = \cos a \cos^2 b + \sin a \sin b \cos b \cos C + \sin b \sin c \cos A;$$

whence, on putting for $\cos^2 b$ its value $1 - \sin^2 b$, dividing by $\sin b$, and transposing, it follows that

$$\cos a \sin b - \sin a \cos b \cos C = \sin c \cos A.$$

Five similar relations can be derived by permuting the letters; and on applying these six relations to the polar triangle, six others can be derived.

To pursue this topic further is beyond the scope of this book, which aims to give little more than the simplest elements of spherical trigonometry and what is absolutely required for the solution of spherical triangles. Those who are interested can refer to the works on spherical trigonometry by M'Clelland and Preston (Macmillan & Co.), Casey (Longmans, Green, & Co.), Bowser (D. C. Heath & Co.), and others.

N.B. *Questions and exercises on Chapter III. will be found on page* 104.

* Instances in which geometrical properties are deduced by means of trigonometry, are given in Art. 27, Art. 36, (Note 2), Art. 38, (Notes 2, 3).

CHAPTER IV.

SOLUTION OF TRIANGLES.

N.B. The student is recommended to work one or two examples in each set in this chapter *before* reading any of the text.

41. Cases for solution. This chapter is concerned with the numerical solution of spherical triangles. In all there are six cases for solution; these correspond respectively to the six cases for construction which were discussed in Arts. 23, 24. In these cases the given parts are as follows:

I. **Three sides.**

II. **Three angles.**

III. **Two sides** and **their included angle.**

IV. **One side** and **its two adjacent angles.**

V. **Two sides** and **the angle opposite one of them.**

VI. **Two angles** and **the side opposite one of them.**

With slight changes the procedure described in Art. 31 is recommended. Figures may be helpful. Of formulas adapted for logarithmic computation, the necessary ones are (1) Art. 36, (4)–(11) Art. 37, and (1)–(4) Art. 38. If the polar triangle is used in finding the solution, then I. and II. constitute one case, likewise III. and IV., and likewise V. and VI.; and the necessary formulas are (1) Art. 36 (4)–(7) *or* (8)–(11) Art. 37, and (1), (2), *or* (3), (4) Art. 38. Cases V. and VI. must be examined as to ambiguity; and accordingly, they give more trouble than the others. *Unless the triangle satisfies the conditions specified in Arts* 15, 17, 24 V., *its solution is impossible.*

Checks. **The results obtained should always be checked.** Delambre's Analogies and formulas which have not been used in the course of the solution, may be used as check formulas.

N.B. Before doing any of the numerical work the student should try to get a clear idea of the figure of the triangle upon a sphere. This geometrical

conception will enable him to make a reasonable estimate of what the results will be; this estimate will help him to detect wild results that may be obtained by making numerical errors. For example, in ABC let $a = 110°$, $b = 114°$, $C = 10°$; and suppose that the result $c = 76°$ presents itself. A person who has drawn a figure of the triangle on a sphere, or one who has geometrical imagination sufficient to give him an idea of the look of the given triangle, will at once see that the result, $c = 76°$, must be wrong. In working spherical triangles it is much better not to proceed blindly.

42. Case I. Given the three sides.

EXAMPLES.

1. In ABC, given $a = 47° 30'$, $b = 55° 40'$, $c = 60° 10'$. Find A, B, C.

Formulas: $$\tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

$$\tan \tfrac{1}{2} A = \frac{\tan r}{\sin(s-a)}, \quad \tan \tfrac{1}{2} B = \frac{\tan r}{\sin(s-b)}, \quad \tan \tfrac{1}{2} C = \frac{\tan r}{\sin(s-c)}.$$

Check: Law of Sines, or Napier's Analogies, or Delambre's Analogies.

Logarithmic formulas:

$$\log \tan^2 r = \log \sin(s-a) + \log \sin(s-b) + \log \sin(s-c) - \log \sin s, \text{ etc.}$$

Check: $\log \sin a - \log \sin A = \log \sin b - \log \sin B = \log \sin c - \log \sin C$.

$a = 47° 30'$	$\log \sin s = 9.99539 - 10$	$\therefore \tfrac{1}{2} A = 28° 16' 2''$
$b = 55° 40'$	$\log \sin(s-a) = 9.74943 - 10$	$\tfrac{1}{2} B = 34° 33' 41.5''$
$c = 60° 10'$	$\log \sin(s-b) = 9.64184 - 10$	$\tfrac{1}{2} C = 39° 29' 12''$
$\therefore 2s = 163° 20'$	$\log \sin(s-c) = 9.56408 - 10$	$\therefore A = 56° 32' 4''$
$s = 81° 40'$	$\therefore \log \tan^2 r = 18.95996 - 20$	$B = 69° \ 7' 23''$
$s - a = 34° 10'$	$\therefore \log \tan r = 9.47998 - 10$	$C = 78° 58' 24''$
$s - b = 26°$	$\therefore \log \tan \tfrac{1}{2} A = 9.73055 - 10$	
$s - c = 21° 30'$	$\log \tan \tfrac{1}{2} B = 9.83814 - 10$	
	$\log \tan \tfrac{1}{2} C = 9.91590 - 10$	

Check:		
$\log \sin a = 9.86763$	$\log \sin b = 9.91686$	$\log \sin c = 9.93826$
$\log \sin A = 9.92128$	$\log \sin B = 9.97051$	$\log \sin C = 9.99191$
9.94635	9.94635	9.94635

NOTE 1. *Directions for the numerical work:* Fill in the first column; turn up the first four logarithms in the second column (since these logarithms are required by the formulas); compute the last five logarithms in the second column according to the formulas (these computations may be made on another paper, if necessary); find the first three angles of the third column by the tables; thence compute A, B, C.

NOTE 2. If only one angle is required, say A, it can be found by one of formulas (4) Art. 37; preferably, the second. Angle A can also be found (without logarithms) by formula (1) Art. 37.

2. Solve ABC, given that $a = 43° 30'$, $b = 72° 24'$, $c = 87° 50'$.
3. Solve ABC, given that $a = 110° 40'$, $b = 45° 10'$, $c = 73° 30'$.
4. Solve ABC, given that $a = 120° 50'$, $b = 98° 40'$, $c = 74° 60'$.
5. Solve PQR, given that $p = 67° 40'$, $q = 47° 20'$, $r = 83° 50'$.

43. Case II. Given the three angles.

Either: Solve the polar triangle by the method used in Case I., and therefrom obtain the parts of the original triangle.

Or: Solve by means of formulas (8)–(11) Art. 37.

EXAMPLES.

Solve ABC, and check the results.

1. Given $A = 74° 40'$, $B = 67° 30'$, $C = 49° 50$.
2. Given $A = 112° 30'$, $B = 83° 40'$, $C = 70° 10'$.
3. Given $A = 130°$, $B = 98°$, $C = 64°$.
4. Given $P = 33° 40'$, $Q = 26° 10'$, $R = 20° 30'$. Find p, q, r.

NOTE. The results may also be checked by solving the examples by both the methods above.

44. Case III. Given two sides and their included angle.

EXAMPLES.

1. In ABC, $a = 64° 24'$, $b = 42° 30'$, $C = 58° 40'$; find A, B, c.

Formulas:

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2} C;$$

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \tfrac{1}{2} C;$$

$$\sin c = \frac{\sin a}{\sin A} \sin C.$$

Checks: Law of Sines, etc.

$C = 58° 40'$	$\log \cot \frac{1}{2} C = 0.25031$	$\therefore \log \tan \frac{1}{2}(A+B) = 0.46743$
$a = 64° 24'$	$\log \sin \frac{1}{2}(a+b) = 9.90490-10$	$\log \tan \frac{1}{2}(A-B) = 9.62405$
$b = 42° 30'$	$\log \cos \frac{1}{2}(a+b) = 9.77490-10$	$\therefore \frac{1}{2}(A+B) = 71° 10' 41''$
$\therefore a+b = 106° 54'$	$\log \sin \frac{1}{2}(a-b) = 9.27864-10$	$\frac{1}{2}(A-B) = 22° 49' 12''$
$a-b = 21° 54'$	$\log \cos \frac{1}{2}(a-b) = 9.99202-10$	$\therefore A = 93° 59' 53''$
$\frac{1}{2} C = 29° 20'$	$\log \sin a = 9.95513-10$	$B = 48° 21' 29''$
$\frac{1}{2}(a+b) = 53° 27'$	$\log \sin C = 9.93154-10$	$\therefore \log \sin A = 9.99894-10$
$\frac{1}{2}(a-b) = 10° 57'$		$\therefore \log \sin c = 9.88773-10$
		$\therefore c = 50° 33' 6''$

NOTE 1. Since $C<A$, then $c<a$; and hence, the acute value of c is taken.

NOTE 2. *Directions for the numerical work:* Fill in the first column; then turn up *all* the logarithms for the second column, these logarithms being required by the formulas; then compute the first two logarithms in the third column, according to the formulas; thence find the corresponding angles, and calculate A and B; turn up $\log \sin A$; compute $\log \sin c$ according to the formula; then find c in the Tables.

NOTE 3. In using formulas involving the difference of two sides or two angles, place the larger side or angle first.

2. Solve ABC, given $a = 93° 20'$, $b = 56° 30'$, $C = 74° 40'$.
3. Solve ABC, given $b = 76° 30'$, $c = 47° 20'$, $A = 92° 30'$.
4. Solve ABC, given $c = 40° 20'$, $a = 100° 30'$, $B = 46° 40'$.
5. Solve PQR, given $q = 76° 30'$, $r = 110° 20'$, $P = 46° 50'$.

45. Case IV. Given one side and its two adjacent angles.

Either: Solve the polar triangle by the method used in Case III.; and therefrom obtain the parts of the original triangle.

Or: Solve by using formulas (1), (2), Art. 38.

EXAMPLES.

1. Solve ABC, given $A = 67° 30'$, $B = 45° 50'$, $c = 74° 20'$.
2. Solve ABC, given $B = 98° 30'$, $C = 67° 20'$, $a = 60° 40'$.
3. Solve ABC, given $C = 110°$, $A = 94°$, $b = 44°$.
4. Solve PQR, given $R = 70° 20'$, $Q = 43° 50'$, $p = 50° 46'$.

46. Case V. Given two sides and the angle opposite one of them.

This is an **ambiguous case,*** since (Art. 24, V.) there may be two solutions. It may be well to examine this case (1) *geometrically,* that is, by an inspection of the figure; (2) *analytically,* that is, by an inspection of the formulas involved in its solution.

(1) *Geometrically.* In Art. 24, V. (Figs. 21, 25) it has been seen that, when two sides and an angle opposite one of them (say, a, b, A) of a triangle ABC are given, there are *two* triangles possible if either of the following sets of conditions holds, viz.:

$$A < 90°,\ a > p,\ a < b, \text{ and } a < 180° - b; \qquad (a)$$

$$A > 90°,\ a < p,\ a > b, \text{ and } a > 180° - b. \qquad (b)$$

* For a detailed discussion of the ambiguous case, see Todhunter, *Spherical Trigonometry*, pp. 53–58; M'Clelland and Preston, *Spherical Trigonometry*, pp. 137–143.

In order that the triangle be *possible*, it is apparent that: *either* $CB = CP$; or, in Fig. 21, $CB > CP$, *i.e.* $\sin CB > \sin CP$, *i.e.* $\sin a > \sin AC \sin CAP$,

$$i.e.\ \sin a > \sin b \sin A;$$

and, in Fig. 25, $CLB < CP$, and $CLB > CP'$, *i.e.* $\sin a > CP'$, *i.e* $\sin a > \sin AC \sin CAP'$,

$$i.e.\ \sin a > \sin b \sin (180^\circ - CAP),\ i.e.\ \sin a > \sin b \sin A.$$

Art. 24 also shows that, when the triangle is possible, there is *one* solution if either of the following sets of conditions holds, viz.:

$$A < 90^\circ,\ a > p,\ a \text{ between } b \text{ and } 180^\circ - b; \qquad (c)$$

$$A > 90^\circ,\ a < p,\ a \text{ between } b \text{ and } 180^\circ - b. \qquad (d)$$

If $CB = CP$, *i.e.* if $a = p$, then there is *one* solution.

Art. 24 also shows that the triangle is impossible if either one of the following sets of conditions holds, viz.:

$$A < 90^\circ,\ a \text{ greater than both } b \text{ and } 180^\circ - b; \qquad (e)$$

$$A > 90^\circ,\ a \text{ less than both } b \text{ and } 180^\circ - b.$$

Since the greater angle is opposite the greater side, B must be such that $A - B$ shall have the same sign as $a - b$.

(2) *Analytically.* The formulas used in solving this case are as follows:

$$\sin B = \frac{\sin b \sin A}{\sin a}, \qquad (1)$$

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}(a - b)} \tan \tfrac{1}{2}(A - B), \quad [\text{or, (4) Art. 38}] \qquad (2)$$

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(a - b). \quad [\text{or, (2) Art. 38}] \qquad (3)$$

Since B is determined from its sine, it may be in either the first or the second quadrant. If $\sin a = \sin b \sin A$, then $B = 90^\circ$. If $\sin a < \sin b \sin A$, then $\sin B > 1$, and B has an impossible value, and, accordingly, the triangle is impossible. [Compare above.] Equation (2) shows that $A - B$ and $a - b$ have the same sign.

Hence, from the *analytical* inspection comes the following **rule:**

If $\sin a < \sin b \sin A$, *there is* **no** *solution; if* $\sin a = \sin b \sin A$, *there is* **one** *solution; if* $\sin a > \sin b \sin A$, *and if* **both** *values of B obtained from* (1) *be such that*

$$A - B \text{ and } a - b \text{ have } \textbf{like} \text{ signs,}$$

there are **two** *solutions; if only* **one** *of the values of B satisfies this condition, there is only* **one** *solution; if* **neither** *of the values of B satisfies this condition, the solution is* **impossible.**

From the *geometrical* inspection comes the following **rule:**

If $\sin a < \sin b \sin A$, *there is* **no** *solution; if* $\sin a = \sin b \sin A$, *there is* **one** *solution; if* $\sin a > \sin b \sin A$, *then:*

When A is less than 90°:

there are **two** *solutions if a is* **less** *than both b and $180° - b$;*
there is **one** *solution if a lies* **between** *b and $180° - b$;*
there is **no** *solution if a is* **greater** *than both b and $180° - b$.*

When A is greater than 90°:

there are **two** *solutions if a is* **greater** *than both b and $180° - b$;*
there is **one** *solution if a lies* **between** *b and $180° - b$;*
there is **no** *solution if a is* **less** *than both b and $180° - b$.*

NOTE 1. The second rule has one advantage over the first, in that it enables one to say, merely on calculating $\sin B$, but without finding B, whether the triangle is ambiguous or not.

NOTE 2. The property observed in Art. 36, Note 2, is frequently used in investigating the ambiguous case.

EXAMPLES.

1. In ABC, $a = 43° 20'$, $b = 48° 30'$, $A = 58° 40'$; find B, C, c.

Formulas:

$$\sin B = \frac{\sin b \sin A}{\sin a}.$$

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(b + a)}{\sin \frac{1}{2}(b - a)} \tan \tfrac{1}{2}(B - A). \qquad \text{[Art. 38, (3)]}$$

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(B + A)}{\sin \frac{1}{2}(B - A)} \tan \tfrac{1}{2}(b - a). \qquad \text{[Art. 38, (1)]}$$

Checks: Formulas (2), (4), Art. 38; or, formulas, Art. 37; or, Delambre's Analogies.

$A = 58^\circ 40'$	$\log \sin A = 9.93154 - 10$	$\therefore\ B + A = 127^\circ 27'$
$a = 43^\circ 20'$	$\log \sin a = 9.83648 - 10$	$B - A = 10^\circ\ 7'$
$b = 48^\circ 30'$	$\log \sin b = 9.87446 - 10$	$\frac{1}{2}(B + A) = 63^\circ 43' 30''$
$\therefore\ b + a = 91^\circ 50'$	$\therefore\ \log \sin B = 9.96952 - 10$	$\frac{1}{2}(B - A) = 5^\circ\ 3' 30''$
$b - a = 5^\circ 10'$	$\therefore\ B = 68^\circ 47'$	$\therefore\ B' + A = 169^\circ 53'$
$\frac{1}{2}(b + a) = 45^\circ 55'$	$B' = 111^\circ 13'$	$B' - A = 52^\circ 33'$
$\frac{1}{2}(b - a) = 2^\circ 35'$	[According to the test for ambiguity.]	$\frac{1}{2}(B' + A) = 84^\circ 56' 30''$
		$\frac{1}{2}(B' - A) = 26^\circ 16' 30''$

In ABC. (*See Fig.* 21, *Art.* 24.) *In AB'C.*

In ABC.	In $AB'C$.
$\log \sin \frac{1}{2}(b + a) = 9.85632 - 10$	As in ABC.
$\log \sin \frac{1}{2}(b - a) = 8.65391 - 10$	
$\log \tan \frac{1}{2}(b - a) = 8.65435 - 10$	
$\log \sin \frac{1}{2}(B + A) = 9.95264 - 10$	$\log \sin \frac{1}{2}(B' + A) = 9.99830 - 10$
$\log \sin \frac{1}{2}(B - A) = 8.94532 - 10$	$\log \sin \frac{1}{2}(B' - A) = 9.64609 - 10$
$\log \tan \frac{1}{2}(B - A) = 8.94702 - 10$	$\log \tan \frac{1}{2}(B' - A) = 9.69345 - 10$
$\therefore\ \log \cot \frac{1}{2} C = 0.14943$	$\therefore\ \log \cot \frac{1}{2} C = 0.89586$
$\log \tan \frac{1}{2} c = 9.66167 - 10$	$\log \tan \frac{1}{2} c = 9.00656 - 10$
$\therefore\ \frac{1}{2} C = 35^\circ 19' 55''.4$, $\frac{1}{2} c = 24^\circ 38' 53''$.	$\therefore\ \frac{1}{2} C = 7^\circ 14' 36''$, $\frac{1}{2} c = 5^\circ 47' 49''$.
$\therefore\ C = 70^\circ 39' 51''$, $c = 49^\circ 17' 46''$.	$\therefore\ C = 14^\circ 29' 12''$, $c = 11^\circ 35' 38''$.

Hence, the solutions are:

$$ABC = 68^\circ 47',\quad ACB = 70^\circ 39' 51'',\quad AB = 49^\circ 17' 46'';$$
$$AB'C = 111^\circ 13',\quad ACB' = 14^\circ 29' 12'',\quad AB' = 11^\circ 35' 38''.$$

NOTE 3. *Directions for the numerical work:* Fill in the first of the three columns; turn up the first three logarithms in the second column, these being required by the first formula; compute $\log \sin B$ according to the first formula; find B in the tables; *decide the question of ambiguity;* fill in the third column (only four lines when the triangle is not ambiguous). Turn up the first six logarithms in the first of the next two columns; compute the next two logarithms according to the formulas; find the corresponding values in the Tables; thence compute C and c. If the case is ambiguous, do the same work for the second triangle.

2. Solve ABC when $a = 56^\circ 40'$, $b = 30^\circ 50'$, $A = 103^\circ 40'$.
3. Solve ABC when $a = 30^\circ 20'$, $b = 46^\circ 30'$, $A = 36^\circ 40'$.
4. Solve ABC when $c = 74^\circ 20'$, $a = 119^\circ 40'$, $C = 88^\circ 30'$.
5. Solve ABC when $b = 30^\circ 10'$, $c = 44^\circ 30'$, $B = 86^\circ 50'$.
6. Solve PQR when $q = 42^\circ 30'$, $r = 46^\circ 50'$, $Q = 56^\circ 30'$.

47. Case VI. Given two angles and the side opposite one of them. This is also an ambiguous case.

Either: Solve the polar triangle by the method used in Case V.; and therefrom obtain the parts of the original triangle.

Or: Solve by using formula (1) Art. 36, and Napier's Analogies.

The first rule (Art. 46) for determining ambiguity suits the case, if a, b, be substituted for A, B, therein. On making use of the polar triangle, it is found that the second rule can be adapted by substituting a, A, B, for A, a, b, respectively.

EXAMPLES.

1. Solve ABC when $A = 108° 40'$, $B = 134° 20'$, $a = 145° 36'$.
2. Solve ABC when $B = 36° 20'$, $C = 46° 30'$, $b = 42° 12'$.
3. Solve ABC when $C = 62° 10'$, $A = 23° 46'$, $c = 33° 50'$.
4. Solve STV when $T = 102° 50'$, $V = 81° 20'$, $t = 124° 30'$.

48. Subsidiary angles. Formulas can sometimes be adapted for logarithmic computation and the triangle solved, by the use of *subsidiary angles.* *For example*, in ABC let a, c, B be known, and b required. (See Fig. 35, Art. 33.)

$$\cos b = \cos a \cos c + \sin a \sin c \cos B \qquad \text{(Art. 36, } B\text{)}$$

$$= \cos c\,(\cos a + \sin a \tan c \cos B).$$

On putting $\tan c \cos B = \tan \phi$, this becomes

$$\cos b = \cos c\,(\cos a + \sin a \tan \phi)$$

$$= \frac{\cos c\,(\cos a \cos \phi + \sin a \sin \phi)}{\cos \phi}$$

$$= \frac{\cos c \cos (a - \phi)}{\cos \phi}.$$

On referring to Fig. 35 it is seen that $BD = \phi$, that $DC = a - \phi$, and $\cos AD = \dfrac{\cos c}{\cos \phi}$; so that solving as above is equivalent to solving the triangle by dividing it into right-angled triangles.

N.B. *Questions and exercises on Chapter IV. will be found on page* 105.

CHAPTER V.

CIRCLES CONNECTED WITH SPHERICAL TRIANGLES.

49. The circumscribing circle. The circle passing through the vertices of a spherical triangle is called the *circumscribing circle,* or *circum-circle,* of the triangle. This circle can be constructed in somewhat the same manner as the circumscribing circle of a plane triangle.

Let ABC (Fig. 41) be a spherical triangle, and let R denote the radius (*i.e.* the polar distance, Art. 6) of its circumscribing circle. Bisect the arcs BC, CA, in L, M, respectively; and at L, M, draw arcs at right angles to BC, CA, respectively. The point O, at which these arcs meet, is the pole of the circumscribing circle.

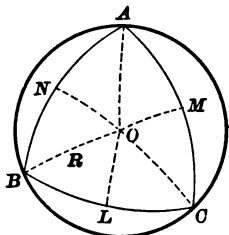



FIG. 41

For, draw OA, OB, OC, arcs of great circles. In the triangles OLB and OLC, $BL = LC$, LO is common, and the angles at L are right angles. Hence, $OB = OC$. In a similar way it can be shown that $OC = OA$. Hence O is the pole of the circumscribing circle.

Join O and N, the middle point of AB; then it is easily shown that ON is at right angles to AB.

In ABC, $\qquad A + B + C = 2S.$

Now (since $OA = OB = OC$),

$$OAB = OBA, \quad OBC = OCB, \quad OCA = OAC.$$

Hence, $OAB + OBC + OAC = S.$

$$\therefore \; OBC = S - (OAB + OAC) = S - A.$$

In the right-angled triangle OBL,

$$\tan OB = \frac{\tan BL}{\cos OBL}. \qquad \text{[Art. 26, Eq. (3)]}$$

$$i.e.\ \tan R = \frac{\tan \frac{1}{2} a}{\cos (S - A)}. \qquad (1)$$

Similarly, $\tan R = \dfrac{\tan \frac{1}{2} b}{\cos (S - B)}$, $\tan R = \dfrac{\tan \frac{1}{2} c}{\cos (S - C)}$.

On substituting in (1) the value of $\tan \frac{1}{2} a$ in relation (8) Art. 37, equation (1) becomes

$$\tan R = \sqrt{\frac{-\cos S}{\cos (S - A) \cos (S - B) \cos (S - C)}}. \qquad (2)$$

NOTE 1. Compare (1) with the corresponding case in plane triangles (*Plane. Trig.*, Art. 68). (In plane triangles, $S = 90°$, and, hence, $\cos (S - A) = \sin A$.)

NOTE 2. On putting $N = \sqrt{-\cos S \cos(S - A) \cos(S - B) \cos(S - C)}$,

$$\tan R = -\frac{\cos S}{N}.$$

50. The inscribed circle. The circle which touches each of the sides of a spherical triangle is called the *inscribed circle,* or *in-circle,* of the triangle. This circle can be constructed in somewhat the same mannér as the inscribed circle of a plane triangle.

Let ABC be a spherical triangle, and let r denote the radius (*i.e.* the polar distance) of its inscribed circle. Bisect angles A, B, by arcs of great circles, and let these arcs meet at O. Draw OL, OM, ON, at right angles to BC, CA, AB, respectively.

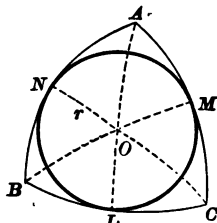
FIG. 42

In the triangles OAM and OAN, the angles at A are equal, the angles at N and M are right angles, and the side OA is common. Hence these triangles are symmetrical, and $OM = ON$. Similarly it can be shown that $ON = OL$. Hence O is the pole of the circle inscribed in ABC.

Since the triangles OAM and OAN are equal, $AM = AN$. Similarly, $BN = BL$, and $CL = CM$.

Now $\qquad AB + BC + CA = 2s;$

hence $\qquad AN + BL + CL = s.$

$$\therefore AN = s - (BL + LC) = s - a.$$

In the right-angled triangle AON,

$$\tan ON = \tan OAN \sin AN. \qquad \text{[Art. 26, (4)]}$$

$$\therefore\ \tan r = \tan \tfrac{1}{2} A \sin (s-a). \qquad (1)$$

Similarly, $\tan r = \tan \frac{1}{2} B \sin (s-b)$; $\tan r = \tan \frac{1}{2} C \sin (s-c)$.

On substituting in (1) the value of $\tan \frac{1}{2} A$ in (4) Art. 37, equation (1) becomes

$$\tan r = \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}}. \qquad (2)$$

On putting $\quad n = \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}, \qquad (3)$

$$\tan r = \frac{n}{\sin s}. \qquad (4)$$

NOTE 1. Compare (1) with *Plane Trigonometry*, Art. 69, Note; (2) with Art. 69, (3); n with S, Art. 66, (3); (4) with (3) Art. 69.

51. Escribed circles. A circle which touches a side of a spherical triangle, and the other two sides produced (that is, which is inscribed in a *co-lunar* triangle), is an *escribed circle*, or an *excircle*, of the triangle. There are three ex-circles, one corresponding to each side of the triangle.

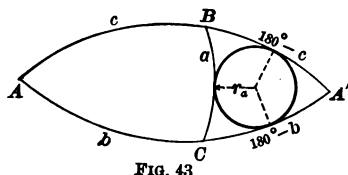



FIG. 43

Let ABC be a spherical triangle; and let the radii of the escribed circles, touching a, b, c, respectively, be denoted by r_a, r_b, r_c, respectively. Complete the lune whose angle is A. The escribed circle which touches a is the inscribed circle of the co-lunar triangle $A'BC$. Hence [Art. 50, (1)],

$$\tan r_a = \tan \tfrac{1}{2} A' \sin \tfrac{1}{2} \left[(a + \overline{180^\circ - b} + \overline{180^\circ - c}) - 2a \right];$$

i.e.

$$\tan r_a = \tan \tfrac{1}{2} A \sin s. \qquad (1)$$

Similarly, $\tan r_b = \tan \frac{1}{2} B \sin s$; $\tan r_c = \tan \frac{1}{2} C \sin s$.

On substituting for $\tan \frac{1}{2} A$ its value in (4) Art. 37, equation (1) becomes

$$\tan r_a = \sqrt{\frac{\sin s \sin (s-b) \sin (s-c)}{\sin (s-a)}}, \quad (2)$$

i.e. $$\mathbf{\tan r_a = \frac{n}{\sin (s-a)}}. \quad \text{[Art. 50, (3)]} \quad (3)$$

Similarly, $\tan r_b = \dfrac{n}{\sin (s-b)}$; $\tan r_c = \dfrac{n}{\sin (s-c)}$.

NOTE. Compare (3) with the corresponding result in *Plane Trigonometry*, Art. 70.

Some other relations between the sides and angles of a spherical triangle and the radii of the circles connected with it, are indicated in the exercises at the end of the book.

Ex. Find the radii of the circumscribing, inscribed, and escribed circles of some of the triangles in Chapters II., IV.

N.B. *For questions and exercises on Chapter V., see page* 107.

CHAPTER VI.

AREAS AND VOLUMES CONNECTED WITH SPHERES.

52. Preliminary propositions.

a. The lateral area of a frustum of a regular pyramid is equal to the product of the slant height of the frustum and half the sum of the perimeters of its bases.

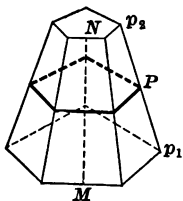
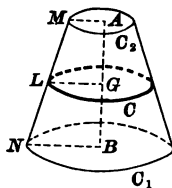
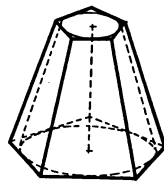



FIG. 44 FIG. 45 FIG. 46

The student can easily prove this (Fig. 44). It should be noted that the half sum of the perimeters of the bases of the frustum is equal to the perimeter of the section which is parallel to the bases and midway between them.

In symbols: If p_1, p_2, P, are the perimeters of the bases and the middle section of the frustum, and MN is its slant height, then

$$\text{lateral area of frustum} = \tfrac{1}{2}\, MN\,(p_1 + p_2) = MN \cdot P.$$

b. The lateral area of a frustum of a cone of revolution is equal to the product of the slant height of the frustum and half the sum of the circumferences of its bases.

[*Suggestion for proof:* If the number of the lateral faces of a frustum of a regular pyramid be indefinitely increased and each face be indefinitely decreased, then this frustum approaches the frustum of a cone of revolution as a limit (see Fig. 46). Accordingly, Proposition (*b*) follows at once from (*a*)]. It should be

noted that half the sum of the circumferences of the bases of the frustum is equal to the circumference of the section which is parallel to the bases and midway between them.

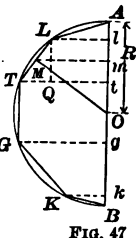
In symbols: If C_1, C_2, C (Fig. 45) are the circumferences of the bases and the middle section of the frustum, and MN is its slant height, then lateral area of frustum

$$= \frac{1}{2} MN\ (C_1 + C_2) = MN \cdot C = 2\ \pi LG \cdot MN.$$

NOTE. The lateral surface of the frustum of the cone (Fig. 45) can be generated by the revolution of the line MN about the line AB which is in the same plane with MN.

53. To find the area of a sphere. The surface of a sphere can be generated by the revolution of a semicircle about its diameter. For example, the semicircle $ATKB$ of radius R on revolving about its diameter AB, will describe the surface of a sphere of radius OA.

FIG. 47

Let a polygon $ALTGKB$ be inscribed in this semicircle. At M, the middle point of one of the chords LT, draw MO at right angles to LT. By geometry, MO will meet AB at O, the middle point of AB. Project LT on AB, the projection being lt; draw LQ at right angles to Tt.

By Art. 52. *b*, the area generated by LT in its revolution about AB

$$= 2\ \pi Mm \cdot LT. \qquad (1)$$

Since the angles of the triangle LTQ are respectively equal to the angles of OMm, these triangles are similar; accordingly,

$$LT : LQ = OM : Mm.$$

$$\therefore Mm \cdot LT = LQ \cdot OM = lt \cdot OM.$$

Hence, from (1), area generated by $LT = 2\ \pi OM \cdot lt. \qquad (2)$

In words: When a chord of a semicircle revolves about the diameter, the area generated is equal to $2\ \pi$ times the product of the length of the perpendicular from the centre to the chord, and the projection of the chord upon the diameter.

∴ The area of the surface generated by the revolution of the polygon $ALTGKB$

$$= 2\pi \times (\text{perpendicular on } AL \text{ from } O) \times Al$$
$$+ 2\pi \times (\text{perpendicular on } LT \text{ from } O) \times lt$$
$$+ 2\pi \times (\text{perpendicular on } TG \text{ from } O) \times tg$$
$$+ 2\pi \times (\text{perpendicular on } GK \text{ from } O) \times gk$$
$$+ 2\pi \times (\text{perpendicular on } KB \text{ from } O) \times kB.$$

If the number of sides in the polygon inscribed in the semicircle is indefinitely increased and each side is indefinitely decreased, then the broken line $ALTGKB$ approaches the semicircle as a limit, and each of the perpendiculars drawn from O to the middle points of the chords approaches R as a limit; while the sum of the projections of the chords remains equal to AB, the diameter of the circle. Hence, area of surface generated by revolution of semicircle $AGB = 2\pi \cdot R \cdot 2R$;

$$\textit{i.e.}\text{ area of surface of sphere of radius } R = 4\pi R^2.$$

In words: The area of the surface of a sphere is four times the area of a great circle of the sphere.

Definition. A **zone** of a sphere is a portion of the surface included between two parallel planes, or, what comes to the same thing, is the portion of the surface included between two circles which have common poles; for example, the surface between the parallels of 30° N. latitude and 50° N. latitude.

The area of a zone. An infinite number of chords can be inscribed in the arc LT (Fig. 47). By reasoning similar to that employed above, it can be shown that

$$\text{area of surface generated by arc } LT = 2\pi R \cdot lt.$$

∴ *The area of a spherical zone is equal to the product of the length of a great circle of the sphere and the height of the zone.*

It follows that on a sphere or on equal spheres the areas of zones of equal heights are equal.

EXAMPLES.

1. Find the area of a sphere of radius 15 inches.

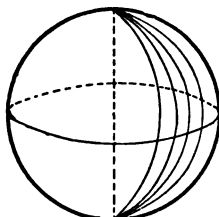
2. Find the surface of a spherical zone of height 2.5 inches on a sphere of diameter 50 inches.

3. Find the convex surface of a spherical segment of height 4.5 inches on a sphere of diameter 7 feet. [See definition, Art. 63.]

4. Suppose that the earth is a sphere whose radius is 3960 miles; find the area of the surface included between the North Pole and the parallel of 80° N. latitude; between the parallels of 49° N. and 50° N.; between 5° N. and 5° S.

54. Lunes. *Definition.* The spherical surface bounded by two halves of great-circles is called a *lune; e.g.* the surface between two meridians. The *angle of the lune* is the angle between the two semicircles; thus the angle of the lune between the meridians 70° W. and 80° W. is 10°.

FIG. 48

Proposition. On the same circle or on equal circles the areas of lunes are proportional to their angles. This can be proved by a method similar to that which is used in proving that the angles at the centre of a circle are proportional to the arcs subtended by them.

55. A spherical degree defined. From the proposition in Art. 54 it follows that the area of a lune is to the area of the surface of the sphere as the angle of the lune is to four right angles. That is,

$$\text{area of lune of angle } A^\circ : \text{area of sphere} = A^\circ : 360^\circ.$$

Hence,
$$\text{area of lune of angle } 1^\circ = \frac{\text{area of sphere}}{360}.$$

Let a great circle be drawn about one of the vertices of a lune of angle 1° as a pole. The lune is then divided into two equal birectangular triangles; accordingly, each triangle contains $(\frac{1}{720})$th of the surface of the sphere, or $(\frac{1}{360})$th of the surface of the hemisphere. *The surface of each such triangle is called* **a spherical degree.**

For example, the part of the surface of a globe bounded by the meridians 43° W. and 63° W. longitude and the equator, contains 20 spherical degrees; the lune bounded by these meridians contains 40 spherical degrees.

A lune of angle $A°$ contains $2A$ spherical degrees.

The passage from spherical degrees of surface to the ordinary measure (of the area) of the surface is easily effected when the radius of the sphere is given.

A spherical degree $= (\frac{1}{720})$*th part of the surface of a sphere;*

hence, **on a sphere of radius r,**

a spherical degree contains $\frac{4\pi r^2}{720}$, *i.e.* $\frac{\pi r^2}{180}$ **square units of area.**

Thus,

area of a lune of angle 20° on a sphere of radius $r = \frac{40\pi r^2}{180} = \frac{2}{9}\pi r^2.$

EXAMPLES.

1. Find the area of a lune of angle 10° on a sphere of radius 2 feet.

2. Find the area of a lune of angle 37° 30′ on a sphere of radius 7 feet.

3. Find the area between the meridians 77° W. and 83° 20′ W.; and the area between the meridians 174° 20′ W. and 158° 35′ E. (Radius of earth = 3960 miles.) [Express areas in spherical degrees and in square miles.]

56. Spherical excess of a triangle. The sum of the angles of a plane triangle is always *equal to* 180°; the sum of the angles of a spherical triangle is always *greater than* 180° (Art. 17). The difference between the latter sum and 180° is called *the spherical excess* of the triangle. (This *excess* is due to the fact that the triangle is spherical and not plane; hence the excess is called *spherical.*) For example, in the triangle bounded by the meridians 47° W. and 48° W. longitude and the equator, the sum of the angles is 181°; and, accordingly, the spherical excess is 1°. In the triangle bounded by the meridians 43° W. and 63° W. and the equator the sum of the angles is 200°, and the spherical excess is 20°; in the spherical triangle having angles 50°, 65°, 125°, the spherical excess is $(50° + 65° + 125° - 180°)$, *i.e.* 60°.

If E denote the number of *degrees* in the spherical excess, and E_r denote the number of *radians* therein, then

in a triangle ABC, $\quad E^\circ = A^\circ + B^\circ + C^\circ - 180^\circ;$ (1)

and [*Plane Trigonometry*, Art. 73, (7)],

$$E_r = \left(\frac{A + B + C - 180}{180}\right)\pi. \quad (2)$$

Ex. Find the spherical excess (in *degrees* and in *radians*) of the triangles described in Art. 42, Exs. 1, 2, 3; Art. 43, Exs. 1, 2; Art. 44, Exs. 1, 2, 3; Art. 45, Exs. 1, 2; Art. 46, Exs. 1, 2, 3; Art. 47, Exs. 1, 2.

57. The area of a spherical triangle.

Proposition: The number of spherical degrees (of surface) *in a spherical triangle is equal to the number of* (angular) *degrees in its spherical excess.**

Let ABC be a spherical triangle whose spherical excess is E°; then area ABC is equal to E spherical degrees. Complete the great circle $BCB'C'$, and produce the arcs BA, CA to meet this circle in B', C', respectively. Complete the great circles $BAB'B$ and $ACA'C'$. The triangle $AB'C'$ is equal to the triangle $A'BC$. For,

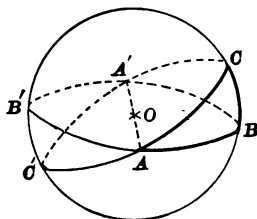



FIG. 49

$$B'A = 180^\circ - AB = BA',$$

$$C'A = 180^\circ - AC = CA',$$

$$C'B' = 180^\circ - B'C = CB.$$

Hence, *in area*, $ABC + AB'C' = \text{lune } ACA'BA;$

also $\qquad ABC + AB'C = \text{lune } BCB'AB;$

and $\qquad ABC + ABC' = \text{lune } CBC'AC.$

* This proposition is sometimes stated thus: *The area of a triangle is equal to its spherical excess;* but this enunciation is rather slipshod.

Hence, on addition,

$$2\,ABC + (ABC + AB'C' + AB'C + ABC') = \text{lune } A + \text{lune } B + \text{lune } C;$$

$$\therefore \quad 2\,ABC = \text{lune } A + \text{lune } B + \text{lune } C - \text{hemisphere.}$$

$$\therefore \text{ (by Art. 55) } 2\,ABC = (2\,A + 2\,B + 2\,C - 360) \text{ spherical degrees.}$$

$$\therefore \quad ABC = (A + B + C - 180) \text{ spherical degrees}$$
$$= E \text{ spherical degrees.}$$

Since (Art. 55) a spherical degree on a sphere of radius r contains $\frac{1}{180}\pi r^2$ square units of area, then, on this sphere,

$$\textbf{area } ABC = \frac{A + B + C - 180}{180}\pi r^2 * = \frac{E}{180}\pi r^2, \qquad (1)$$

$$= E_r r^2, \qquad \text{[Art. 56 (2)]} \qquad (2)$$

in which E denotes the number of degrees, and E_r denotes the number of radians in the spherical excess.

Hence, in order to find the area of a triangle, find the angles, calculate the spherical excess in degrees or radians, and use one of formulas (1), (2).

NOTE. It should be observed that [from Art. 14, Art. 56 (1), and the proposition above], *the number of spherical degrees* contained in the area subtended on a spherical surface by a solid angle at the centre of the sphere, *remains the same*, however the radius may vary. On the other hand, by (1) and (2), *the number of square units* in the subtended area *varies* as the square of the radius.

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* This expression for the area of a spherical triangle was first given in 1629 by Albert Girard (1590–1634) (see *Plane Trigonometry*, pp. 22, 167); and it is often called *Girard's Theorem*. The method of proof used above was invented by John Wallis (1616–1703) professor of geometry at Oxford. (See Wallis, *Works*, Vol. II., p. 875.)

It follows from (1) that

$$\text{area } ABC : 2\,\pi R^2 = E^\circ : 360^\circ.$$

Hence, the above proposition may be expressed thus: *The area of a spherical triangle is to the surface of the hemisphere as the excess of its three angles above two right angles is to four right angles.*

EXAMPLES.

Find the areas of the following triangles (see examples, Art. 56):

1. Those described in Art. 42, Exs. 1, 2, 3, when on a sphere of radius 10 feet.

2. Those described in Art. 43, Exs. 1, 2, when on a sphere of radius 25 inches.

3. Those described in Art. 44, Exs. 1, 2, 3, when on a sphere of radius 30 yards.

4. Those described in Art. 45, Exs. 1, 2, when on a sphere of radius 4 feet.

5. Those described in Art. 46, Exs. 1, 2, 3, when on a sphere of radius 18 inches.

6. Those described in Art. 47, Exs. 1, 2, when on a sphere of radius 3960 miles.

58. Formulas for the spherical excess (E°) of a triangle. • Since, in a spherical triangle ABC, $E^\circ = A^\circ + B^\circ + C^\circ - 180^\circ$, and since there are many relations between the sides and angles of a triangle, it may be expected that there can be many formulas for the spherical excess; and, accordingly, for the area of a spherical triangle. [It will be remembered that there are several formulas for the area of a plane triangle (*Plane Trigonometry*, Art. 66).] Following are some of the most important of these (the deduction of some of them is given in Note B):

A. The spherical excess in terms of the three sides.

(*a*) *L'Huillier's* formula:

$$\tan \tfrac{1}{4} E^\circ = \sqrt{\tan \tfrac{1}{2} s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)}.$$

(*b*) *Cagnoli's* formula: $\sin \tfrac{1}{2} E^\circ = \dfrac{n}{2 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}$,

in which $$n = \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}.$$

(*c*) *De Gua's* formula*: $\cot \tfrac{1}{2} E^\circ = \dfrac{1 + \cos a + \cos b + \cos c}{2n}$.†

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* *Simon L'Huillier* (1750–1810), a Swiss mathematician and philosopher; *Antoine Cagnoli* (1743–1816), an Italian astronomer; *L'abbé Jean Paul de Gua* (1712–1786), a French philosopher.

† For the deduction of this formula see Chauvenet, *Trigonometry*, p. 230, and Crawley, *Trigonometry*, p. 166.

B. The spherical excess in terms of two sides and their included angle.

$$(d)\ \tan \tfrac{1}{2} E^\circ = \frac{\tan \frac{1}{2} a \tan \frac{1}{2} b \sin C}{1 + \tan \frac{1}{2} a \tan \frac{1}{2} b \cos C};$$

$$(e)\ \cot \tfrac{1}{2} E^\circ = \frac{\cot \frac{1}{2} a \cot \frac{1}{2} b + \cos C}{\sin C}.$$

Ex. By these formulas find the spherical excess of some of the triangles referred to in Ex. 1, Art. 56.

59. *a.* The number of spherical degrees in any figure on a sphere, whatever may be its boundary, is the ratio of the area of the figure to the area of a spherical degree, that is, to $(\frac{1}{360})$th part of the area of the hemisphere (Art. 55). Thus, on a sphere of radius r, if A denotes the area of the figure, and E the number of spherical degrees therein, then, since area of a hemisphere $= 2\pi r^2$,

$$E = A : \frac{1}{360} \text{ of } 2\pi r^2 = \frac{180 A}{\pi r^2}. \tag{1}$$

[Compare Art. 57 (1), Art. 59 (2).]

The plane angle E° may be called the *spherical excess of the figure.* For example, the spherical excess of a lune of angle A° is $2A^\circ$.

***b.* The spherical excess of a** (*non-re-entrant*) **spherical polygon.** On drawing diagonals from any vertex of a polygon of n sides to the other vertices, it will be seen that the polygon is divided into $n - 2$ triangles. The sum of the angles of all these triangles is the same as the sum of the angles of the polygon. Hence,

spherical excess (E°) *of polygon of n sides*

$$= \textit{sum of angles} - (n-2)\,180^\circ.$$

If the radius of the sphere is r, then (Art. 57)

$$\textit{area of the polygon} = \frac{E}{180}\pi r^2. \tag{2}$$

60. Given the area of a figure: to find its spherical excess. More fully: *To find the spherical excess of a figure on a sphere when the area of the figure is given in square units.*

Let r denote the radius of the sphere, A the area of the figure, E the number of degrees, n the number of seconds, and E_r the number of radians, in its spherical excess. Then, by (1) Art. 59,

$$E = \frac{180\,A}{\pi r^2}. \tag{1}$$

$$\therefore\ n = 3600\,E = 206265\frac{A}{r^2}. \tag{2}$$

Now $$1^\circ = \frac{\pi}{180} \text{ radians};$$

hence $$E^\circ = \frac{\pi}{180}E \text{ radians}$$

$$= \frac{A}{r^2} \text{ radians.} \qquad \text{[by (1)]}$$

$$\therefore\ E_r = \frac{A}{r^2}. \tag{3}$$

A particular application of (2) can be made to the following problem, viz.: *The area of a spherical triangle on the earth's surface being known, to derive a formula for computing the spherical excess.*

The length of a degree on the earth's surface is found to be 365155 feet. Accordingly,

$$R \text{ (the radius of the earth)} = \frac{365155 \times 180}{\pi} \text{ feet.} \tag{4}$$

From (2), $$\log n = \log A + \log 206265 - 2\log R. \tag{5}$$

On expressing A in square feet, and substituting in (5) the value of R in (4), there is obtained,

$$\log n = \log A - 9.3267737. \tag{6}$$

Formula (6) is called *Roy's Rule*, as it was used by General William Roy (1726–1790) in the Trigonometrical Survey of the British Isles.* The area of the spherical triangle can be approximately determined to a sufficient degree of accuracy.

* The rule should probably be credited to Isaac Dalby (1744–1824), who was mathematical assistant to General Roy from 1787 to 1790, and later became professor of mathematics at the Royal Military College. [See *Phil. Trans.*, vol. 80 (1790).] This was the first practical application of Gerard's theorem (Art. 57).

61. The measure of a solid angle. A *plane angle* can be measured by *any circular arc which it subtends;* and the measure can be expressed in *radians* and in *degrees.* The *radian* (or circular) *measure of an angle* is the number of times any circular arc subtended by it contains the radius (*Plane Trig.*, Art. 73); and the *number of degrees in the angle* is equal to the number of degrees in the subtended circular arc. Thus, the radian measure of an angle of an equiangular triangle is $\frac{1}{3}\pi$, and its degree measure is 60.

A *solid angle* can be measured in a somewhat similar manner, namely, by means of *any spherical surface which it subtends.* What may be called the **spherical measure** *of a solid angle* is the number of times any spherical surface subtended by it contains an area equal to the square on the radius. For example, since the surface of a sphere is equal to $4\pi r^2$, the sum of all the solid angles about any point is 4π. The angle at the corner of a cube subtends one-eighth of the surface of the sphere; accordingly, its spherical measure is $\frac{4\pi r^2}{8} \div r^2$, *i.e.* $\frac{1}{2}\pi$.
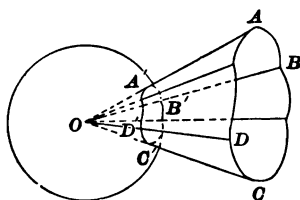
FIG. 50

A solid angle may also be measured in *spherical degrees,* a term that will be explained presently. What may be called the **spherical degree measure** *of a solid angle* (or, the number of spherical degrees in the angle) is a number equal to the number of spherical degrees of area in any spherical surface subtended by the angle. An angle that subtends a spherical degree of surface, contains what may be called *a solid spherical degree.* For example, the sum of all the solid angles about any point is 720 spherical degrees (of angle); the angle at the corner of a cube contains 90 spherical degrees (of angle). Thus the *spherical measure* of the angle at the corner of a cube is $\frac{1}{2}\pi$, and its *spherical degree measure* is 90. On comparing these definitions of solid angular measures with Art. 55 and equations (3) and (1) Art. 60, it is seen that these *measures of solid angles are equal to the measures, in radians and degrees respectively, of the spherical excess of the figures subtended on any sphere by the angle,* when the vertex of the angle is at the centre of the sphere.

NOTE 1. **The term degree.** In geometry and trigonometry the word *degree* is used in connection with *four* very different kinds of quantities; namely, *circular arcs*, *plane angles*, *spherical surfaces*, and *solid angles*.

A *degree of arc*, or *an arcual degree*, is $(\frac{1}{360})$th part of any circle;

A *degree of angle*, or *an angular degree*, is $(\frac{1}{360})$th part of four right angles;

A *degree of surface on a sphere*, or a *spherical degree of surface*, is $(\frac{1}{720})$th part of the surface of any sphere;

A *degree of solid angle*, or a *solid spherical degree*, is $(\frac{1}{720})$th part of the solid angles about any point.

NOTE 2. If two plane angles are equal, they can be superposed, the one on the other. On the other hand, just as two figures on a sphere may be equal in area and differ in every other respect, so two solid angles can be equal in measure and differ in every other respect.

NOTE 3. The following remarks relating to the measurement of solid angles are from Hutton's *Course in Mathematics*, Vol. II., p. 64:

"*Solid angles:* If about the angular point of a solid angle as centre, a sphere be described to radius unity, the portion of its surface intercepted between the planes which contain the solid angle is the *measure of the solid angle*. (This method of estimating the magnitude of solid angles appears to have been first given by Albert Girard in his *Invention Nouvelle en Algebre*, 1629; and it would very naturally suggest itself as one of the simplest applications of his theorem for the spherical excess.)" [Compare *Plane Trigonometry*, p. 126, Note 2.]

Ex. 1. The edge angles of a triedral angle are 74° 40′, 67° 30′, 49° 50′; calculate its spherical degree measure, and its spherical measure. (See Ex. 1, Art. 43.)

Ex. 2. The face angles of a triedral angle are 47° 30′, 55° 40′, 60° 10′; calculate its spherical degree measure, and its spherical measure. (See Ex. 1, Art. 42.)

Ex. 3. Two face angles of a triedral angle are 64° 24′, 42° 30′, and the edge angle between their planes is 58° 40′; calculate its spherical degree measure, and its spherical measure. (See Ex. 1, Art. 44.)

Ex. 4. A face angle of a triedral angle is 74° 20′, and the two adjacent edge angles are 67° 30′ and 45° 50′; calculate its measure. (See Ex. 1, Art. 45.)

Ex. 5. Calculate the spherical degree measure, and the spherical measure, of the solid angles corresponding to the spherical triangles described in Art. 42, Exs. 2, 3; Art. 43, Ex. 2; Art. 44. Exs. 2, 3; Art. 45, Ex. 2; Art 46, Exs. 2, 3; Art. 47, Ex. 2. (See Ex., Art. 56.)

62. The volume of a sphere. In some works on solid geometry and in books on mensuration it is shown that the volume of a pyramid is equal to one third the product of its base and altitude. Now suppose that a polyedron (*i.e.* a solid bounded by plane faces) is circumscribed about a sphere, each of the faces of the polyedron, accordingly, touching the sphere. This polyedron may be regarded as made up of pyramids which have a common vertex (namely, the centre of the sphere), and a common altitude (namely, the radius of the sphere), and which have the faces of the polyedron as bases. Then, R being the radius of the sphere,

$$\text{Vol. of polyedron} = \tfrac{1}{3} R \times (\text{sum of faces of polyedron}). \quad (1)$$

If the number of faces of the polyedron be increased and the area of each face be decreased, then the sum of the faces becomes more nearly equal to the area of the surface of the sphere, and the volume of the polyedron becomes more nearly equal to the volume of the sphere. By increasing the number of faces and decreasing the area of each face, the difference between the sum of the faces of the polyedron and the area of the sphere can be made as small as one please; and, likewise, the difference between the volume of the polyedron and the volume of the sphere can be made as small as one please. In other words:

The area of the surface of the sphere is the limit of the area of the surface of the polyedron, and the volume of the sphere is the limit of the volume of the polyedron, when the faces of the latter are increased without limit, and each face is made to approach zero in area.

Hence, from (1), Vol. of sphere $= \frac{1}{3} R \times$ surface of sphere

$$= \tfrac{4}{3}\pi R^3.* \quad (2)$$

63. Definitions. A **spherical pyramid** is a portion of a sphere bounded by a spherical polygon and the planes of the sides of the polygon. The polygon is called the *base* of the pyramid.

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* For a note concerning the measurement of the circle and the sphere see *Plane Trigonometry*, Art. 72, and Note *C*, p. 171. For the proofs of Archimedes, see T. L. Heath, *The Works of Archimedes edited in modern notation, with introductory chapters* (Cambridge, University Press), pp. 39, 41, 93.

For example, in Fig. 11, Art. 12, $O–ABCD$, $O–ABC$, $O–ABD$, are spherical pyramids; their bases are $ABCD$, ABC, ABD.

A **spherical sector** is the portion of a sphere generated by the revolution of a sector of a circle about any diameter of the circle as axis. For example, in Fig. 47, Art. 53, when the semicircle ATB revolves about AB, each of the circular sectors AOL, LOT, LOK, etc., describes a spherical sector.

A **spherical segment** is the portion of a sphere bounded by two parallel planes and the zone intercepted between them. (One of the planes may be tangent to the sphere.)

64. Volume of a spherical pyramid; of a spherical sector. By reasoning analogous to that in Art. 62, it can be shown that, in a sphere of radius R,

$$\text{vol. of a spherical pyramid} = \tfrac{1}{3} R \times \text{area of its base};$$

$$\text{vol. of a spherical sector} = \tfrac{1}{3} R \times \text{area of its zone}.$$

Since the area of a zone of height $h = 2\,\pi Rh$ (Art. 53),

then $$\text{vol. of spherical sector} = \tfrac{2}{3}\,\pi R^2 h.$$

Thus in Fig. 11, Art. 12,

$$\text{vol. } O–ABCD = \tfrac{1}{3}\, OA \times \text{area } ABCD;$$

in Fig. 47, Art. 53,

vol. of sector described by $AOL = \frac{1}{3}\, OA \times$ area of zone described by arc $AL = \frac{2}{3}\,\pi R^2 \cdot Al$, and

vol. of sector described by $LOT = \frac{1}{3}\, OA \times$ area of zone described by arc $LT = \frac{2}{3}\,\pi R^2 \cdot lt$.

EXAMPLES.

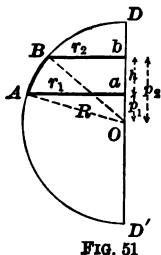
1. Find the volumes of the spherical pyramids whose bases are the triangles described in Art. 57, Exs. 1–6.

2. Find the volumes of the following spherical sectors:

(*a*) The sector whose base is a zone of height 2 inches on a sphere of radius 18 inches.

(*b*) The sector whose base is a zone of height 3 feet on a sphere of radius 12 feet.

65. Volume of a spherical segment. Let AB be an arc of a semicircle of radius R having the diameter DD'. From A, B, draw Aa, Bb, at right angles to DD'. It is required to find the volume of the spherical segment generated by the revolution of $ABba$ about DD'.

FIG. 51

Let h denote the height of the segment, and p_1, p_2, the lengths of the perpendiculars from the centre O to the parallel bases of the segment. On making the revolution of the semicircle DAD', it is seen that

segment generated by $ABba$ = cone generated by BOb + spherical sector generated by AOB − cone generated by AOa.

Now, $$\text{vol. cone generated by } BOb = \tfrac{1}{3}\pi r_2^2 p_2;$$

$$\text{vol. sector generated by } AOB = \tfrac{2}{3}\pi R^2 h; \qquad \text{(Art. 64)}$$

$$\text{vol. cone generated by } AOa = \tfrac{1}{3}\pi r_1^2 p_1.$$

$$\therefore \text{ vol segment} = \tfrac{1}{3}\pi(r_2^2 p_2 + 2R^2 h - r_1^2 p_1). \qquad (1)$$

NOTE. The result (1) can be reduced to various forms. For example, since

$$p_1^2 = R^2 - r_1^2, \; p_2^2 = R^2 - r_2^2, \; p_2 - p_1 = h,$$

then $$\text{vol. segment} = \tfrac{2}{3}\pi R^2(p_2 - p_1) + \tfrac{1}{3}\pi p_2(R^2 - p_2^2) - \tfrac{1}{3}\pi p_1(R^2 - p_1^2)$$

$$= (p_2 - p_1)\pi R^2 - \tfrac{1}{3}\pi(p_2^3 - p_1^3) \qquad (2)$$

$$= \frac{p_2 - p_1}{3}\pi[3R^2 - (p_2^2 + p_2 p_1 + p_1^2)]. \qquad (3)$$

Since $$h = p_2 - p_1, \text{ then } h^2 = p_2^2 - 2p_2 p_1 + p_1^2.$$

$$\therefore p_1 p_2 = \frac{p_2^2 + p_1^2 - h^2}{2}, \text{ and } p_2^2 + p_2 p_1 + p_1^2 = \frac{3}{2}(p_1^2 + p_2^2) - \frac{h^2}{2}.$$

On substituting the last result in (3), expressing p_1^2 and p_2^2 in terms of R, r_1, r_2, and reducing, the following formula is obtained, viz.:

$$\text{vol. segment} = \frac{\pi h}{2}\left(r_1^2 + r_2^2 + \frac{h^2}{3}\right). \qquad (4)$$

EXAMPLES.

1. Show that if (in Fig. 51) angle $AOD = \alpha$, then the volume of the spherical sector generated by AOD is $\frac{2}{3}\pi R^3(1 - \cos\alpha)$.

2. Show that if angle $AOD = \alpha$, then the volume of the segment generated by the revolution of ADa is $\frac{4}{3}\pi R^3 \sin^4 \frac{1}{2}\alpha(1 + 2\cos^2 \frac{1}{2}\alpha)$.

Suggestion. Segment generated by ADa = sector generated by AOD − cone generated by AOa.

3. Find the volume of a spherical segment, the diameters of its ends being 10 and 12 inches, and its height 2 inches.

4. The diameters of the ends of a spherical segment are 8 and 12 inches, and its height is 10 inches. Find its volume.

N.B. *For questions and exercises on Chapter VI., see page* 108.

CHAPTER VII.

PRACTICAL APPLICATIONS.

66. Geographical problem. *To find the distance between two places and the bearing* (*i.e.* the direction) *of each from the other, when their latitudes and longitudes are known.* An interesting application of spherical trigonometry can be made in solving this problem. In the following examples the earth is regarded as spherical, and its radius is taken to be 3960 miles.

EXAMPLES.

1. Find the shortest distance along the earth's surface between Baltimore (lat. 39° 17′ N., long. 76° 37′ W.) and Cape Town (lat. 33° 56′ S., long. 18° 26′ E.).

In Fig. 52 B and C represent Baltimore and Cape Town; EQ is the earth's equator; NGS, NBS, NCS are the meridians of Greenwich, Baltimore, and Cape Town respectively; BC is the great circle arc whose length is required.

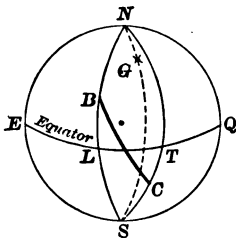



FIG. 52

In the spherical triangle BNC, NB, NC, and BNC are known. For

$$NB = 90° - BL = 90° - 39° 17' = 50° 43'$$
$$NC = 90° + TC = 90° + 33° 56' = 123° 56'$$
$$BNC = BNG + GNC = 76° 37' + 18° 26' = 95° 3'$$

Hence, BC can be determined in degrees by Art. 44; then, the radius of the sphere being given, BC can be determined in miles. The angles NBC, NCB, can also be found.

Answers: $BC = (65° 47' 48'') = 4685.8$ miles; $NBC = 115° 1' 35'$; $NCB = 57° 42' 23''$.

NOTE 1. The *bearing* of one place from a second place is the angle which the great circle arc joining the two places makes with the meridian of the second place. Thus, in Fig. 52 the bearing of Cape Town from Baltimore is the angle NBC, and the bearing of Baltimore from Cape Town is NCB.

Since $NBC = 115° 1' 35''$ the ship *sets out from Baltimore* on a course S. 64° 58′ 25″ E.; since $NCB = 57° 42' 23''$ the ship *approaches Cape Town* on a course S. 57° 42′ 23″ E.

NOTE 2. A ship that sails on a great circle (excepting the equator or a meridian) must be continually changing her course.

2. Find the latitude of the place where BC crosses the meridian 15° W.; also find the bearing of Cape Town from this place.

3. If a vessel sails from Baltimore and keeps constantly on the course (see Ex. 1) S. 64° 58′ 25″ E. (*i.e.* crosses every meridian at the angle 64° 58′ 25″), will she arrive at Cape Town? [*Answer.* No.]

4. What path will the vessel in Ex. 3 make on the sea? *Answer.* A path which is a spiral going round and round the earth and gradually approaching the south pole. This path is called the *loxodrome*, or *rhumb line.*

5. If a person leaves Boston, Mass. (lat. 42° 21′ N., long. 71° 4′ W.), starting due east, and keeps on a great circle: (*a*) Where will he be after he has passed over an arc of 90°, and in what direction will he be going? (*b*) Where will he be after he has passed over an arc of 180°, and in what direction will he be going? (*c*) Where will he be after he has passed over an arc of 270°, and in what direction will he be going? [Solve this example: (1) by spherical geometry; (2) by spherical trigonometry.]

6. What is the distance from New York (40° 43′ N., 74° 0′ W.) to Liverpool (53° 24′ N., 3° 4′ W.)? Find the bearing of each place from the other. In what latitude will a steamer sailing on a great circle from New York to Liverpool cross the meridian of 50° W.; and what will be her course at that point?

N.B. *Check the results in the following exercises:*

7. Find the distance and bearing of Liverpool from Montreal (45° 30′ N., 73° 33′ W.).

8. Find the distance and bearing of Liverpool from Halifax, N. S. (44° 40′ N., 63° 35′ W.).

9. Find the distance and bearing of Santiago de Cuba (20° N., 75° 50′ W.) from Rio de Janeiro (22° 54′ S., 43° 8′ W.).

10. Find the distance and bearing of San Francisco (37° 47′ 55″ N., 122° 24′ 32″ W.) from New York.

11. Find the distance of Victoria, B. C. (48° 25′ N., 123° 23′ W.) from Sydney, N. S. W. (33° 52′ S., 151° 13′ E.); and the bearing of each place from the other.

12. Find the distances between the following places: (*a*) San Francisco and Honolulu; (*b*) Cape Town and Cairo; (*c*) Honolulu and Manila; (*d*) Victoria, B. C., and Tokio.

13. Find the distances between other places, and their bearings from each other.

APPLICATIONS TO ASTRONOMY.

N.B. *In connection with his study of the following articles the student should consult some elementary text-book on astronomy. The numerical examples given here will supplement his outside reading on spherical astronomy.*

67. One of the most important applications of spherical trigonometry is to astronomy. Trigonometry was invented to aid astronomy, and for centuries was studied as an adjunct of the latter subject. (See *Plane Trigonometry*, pp. 165, 166.) A few of the simplest problems of spherical astronomy are introduced in Arts. 73, 74. In order to understand these problems a clear conception of a few astronomical terms and principles is necessary. These terms are explained in Arts. 68–72.

68. The celestial sphere. To a person on the surface of the earth, the sky above is like a great hemispherical bowl with himself at the centre. The stars seem to move from east to west across the spherical sky in parallel circles whose axis is the earth's polar axis prolonged. Each star makes a complete revolution about this axis in 23 hours 56 minutes ordinary clock time. The stars appear never to change their positions with reference to one another, being in this respect like places on the earth's surface.* Another way of describing the relations of the earth and the enveloping sky, is to say that the whole sky is turning, like an immense crystal sphere, about an axis which is the earth's polar axis prolonged, the motion being from east to west. The stars keep the same positions with respect to one another, and, accordingly, appear to be attached to the surface of the sphere. As the sphere turns, the stars fixed in it appear to trace parallel circles

* The positions of some of the stars suffer a very slight change which is perceptible in the course of centuries.

about the axis. The sphere turns completely in 23 hours 56 minutes ordinary clock time.* The *stars all seem to be at the same distance* from the observer because his eyes can judge their *directions only,* and not their *distances.*

The following considerations will show that it is natural enough for an observer on the earth to think that he is always *at the centre* of the sphere on which the stars appear to be. When a person changes his position, the direction of an object at which he is looking changes also, unless he moves directly towards or away from the object. For instance, from a certain point a tree may be in an easterly direction, and when the observer moves a little way the tree may be in a southeasterly direction. Moreover, the further away an object is, the less will be the change in its direction caused by any particular change in the observer's position. Thus, if a person is near a tree, a few steps on his part may change the direction of the tree from east to southeast, but if he is five miles from the tree, an equal number of steps taken by him will make very little difference in the direction of the tree. Now the earth's mean distance from the sun is about 93,000,000 miles. Hence, an observer who now looks at the stars from a certain position, in about six months from now will look at them from a point 186,000,000 miles distant from his present position.† Astronomers have succeeded in a few instances in determining the distances of the stars from the earth.‡ It has been found that the nearest star yet known, *Alpha Centauri,* is so far away that the *change in its direction* from the centre of the earth, due to the change of position of 186,000,000 miles on the part of the earth, *is less* than the change in the direction of an object 3¼ miles away when the observer moves his head a couple of inches at right angles to the line of sight. This being so in the case of the sun's *nearest* stellar neighbour, it is natural for an observer on the earth to think that he is always at the centre of the great sphere on which the stars appear to be; and it is perfectly proper

* The student probably knows that the apparent turning of the spherical sky from east to west about an axis which is the earth's polar axis prolonged, is really due to the rotation of the earth in an opposite direction. The observer is not conscious of any motion of the earth, and thinks that the sky with its bright points is revolving about the earth from east to west, while all the time the sky is motionless, and the earth is turning under it from west to east. Just as to a person in a swiftly moving train the objects outside seem to be rushing by him while the train appears to be at rest.

† This, moreover, does not take any account of the motion of the sun with his system through space.

‡ The first stellar distance determined was that of *61 Cygni* by Friedrich Wilhelm Bessel (1784–1846), one of the greatest of German astronomers, in 1838. Since then the distances of about 100 stars have been measured; about 50 of these distances are regarded as reliably determined.

for him to act in accordance with this notion when he makes astronomical observations and deductions.*

The sphere on which the stars appear to move in parallel circles, or, what comes to the same thing, the sphere which appears to have the stars attached to it and to revolve about the earth's polar axis prolonged, is called **the celestial sphere.**

69. Points and lines of reference on the celestial sphere. There will now be shown some methods for indicating the positions of the heavenly bodies on the celestial sphere — *their positions with respect to the observer* and *their positions with respect to one another.*

The positions of places on the terrestrial sphere are described by means of certain points and great circles on the sphere. There are various pairs of circles which are used for reference; for example, the equator (whose poles are the north and south poles of the earth) and the meridian passing through the Royal Observatory at Greenwich, the equator and the meridian passing through the observatory at Washington, etc. It will be observed that in each case *the reference circles are at right angles to each other, and, accordingly, each of them passes through the poles of the other.*

In an analogous manner the positions of bodies on the celestial sphere are described by means of, or by reference to, certain points and great circles on that sphere. There are *four* different systems of circles of reference. As in the case of the terrestrial sphere, each system consists of two circles, each of which passes through the pole of the other, and, accordingly, is at right angles to the other. Two of these systems are described in Arts. 70, 71, a third in Art. 76, and the fourth in Art. 77. A point which will be referred to in these systems is the **north celestial pole.** This is the point where the earth's axis, if prolonged, would pierce the celestial sphere. It is near the pole star, being about $1\frac{1}{4}°$ from it.

* ". . . imagine the entire solar system as represented by a tiny circle the size of the dot over this letter *i*." (Neptune the outermost planet known of the solar system is 2790 millions of miles from the sun; *i.e.* 30 times as far as the earth.) "Even the sun itself, on this exceedingly reduced scale, could not be detected with the most powerful microscope ever made. But on the same scale the vast circle centred at the sun and reaching to *Alpha Centauri* would be represented by the largest circle which could be drawn on the floor of a room 10 feet square." (Todd, *New Astronomy*, p. 438.)

70. The horizon system: Positions described by altitude and azimuth. For any place on the earth's surface, the point at which the plumb line extended upwards meets the celestial sphere is called **the zenith**; the diametrically opposite point is called **the nadir**. If a plane perpendicular to the plumb line be passed either immediately beneath the observer's feet, or through the centre of the earth, about 4000 miles below him, then the intersection of this plane with the celestial sphere is called **the horizon**. (Since the earth is so small and so far away from even the nearest star, two parallel planes 4000 miles apart and passing through the earth will appear, to a terrestrial observer, to intersect the celestial sphere in the same great circle.)

Great circles passing through the zenith are perpendicular to the horizon; they are called *vertical circles*. The *north point of the horizon* is the point which is directly north from the observer. It is where the vertical circle passing through the north pole intersects the horizon. *This circle which passes through the zenith and the pole is called* **the meridian** *of the observer. The horizon and the meridian* are the reference circles in the horizon system

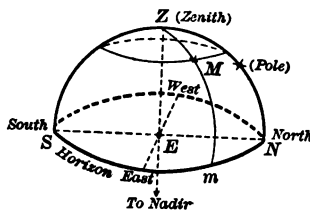



FIG. 53

The **altitude** (denoted by h) of a heavenly body is its angular distance above the horizon. Thus the altitude of M (Fig. 53, in which E is the earth and Z the zenith of the place of observation) is Mm. The altitude of the zenith is 90°. The distance of a star from the zenith is called its *zenith distance;* this is obviously the complement of the altitude.

The **azimuth** (denoted by A) of a heavenly body is the angle between its vertical circle and the meridian. This angle is measured usually along the horizon from the south point in the direction of the west point, to the foot of the star's vertical circle. Thus in Fig. 53 the azimuth of M is $180° + NZm$, which is measured by the arc $180° + Nm$ on the horizon.

NOTE. Any two points on the earth's surface have different zeniths. Hence, the above system of describing positions on the celestial sphere is peculiarly *local*. Moreover, a star rises in the eastern part of the horizon

(altitude zero), mounts higher in the sky until it reaches the observer's meridian, then sinks towards, and sets in, the west; it is, accordingly, continually changing its altitude and azimuth.

71. The equator system: Positions described by declination and hour angle. The north celestial pole is the principal point of this system. **The celestial equator** is the great circle of which that point is the pole; it is evidently the projection of the earth's equator upon the celestial sphere. *The celestial equator and the meridian of the observer* are the reference circles in the system now being described. In Fig. 54, P is the north celestial pole, S the south celestial pole, EQ the celestial equator; also, HR is the horizon and Z the zenith for some particular place on the earth's surface. As said in Art. 68, the stars move in parallel circles whose axis is PS; these circles are, accordingly, parallel to the equator EQ. The angular distance of a star from the equator is called the **declination** (denoted by D or δ) of the star; *north* (or +) *declination* when the star is north of the equator, and *south* (or −) *declination* when the star is south. Thus the declination of S_3 is S_3s_3. The angular distance of a star from the north pole is called its *north polar distance;* this is evidently the complement of the star's declination.*

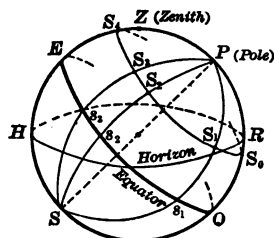



FIG. 54

In 24 (sidereal) hours a star appears to make a complete revolution (*i.e.* to pass over 360°) about the celestial polar axis; hence, *the star passes over 15° in 1 hour.*† The great circles passing through the poles are called *hour circles.* Thus PS_3S is the hour circle of S_3. **The hour angle** (denoted by $H.A.$) of a star is the angle between the meridian of the observer and the hour circle of the

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* The declination of the stars change by an exceedingly small amount in the course of a year.

† The interval of time between two successive passages of the observer's meridian by the sun (*i.e.* from noon to noon) is about 4 minutes longer than the interval of time between two successive passages of the meridian by any particular star. (This difference is due to the yearly revolution of the earth about the sun. See text-books on astronomy.) The second interval is called a sidereal day; it is divided into 24 sidereal hours.

star. This angle is measured *towards the west.* Thus, suppose that a star is on the meridian at S_4; its hour angle is then zero. Twelve hours later the star will be at S_0, and will have an hour angle 180°. After a while it will be at S_1, just rising above the horizon, and its hour angle will be $180° + S_0PS_1$; later it will be at S_3, having the hour angle $180° + S_0PS_3$; later still it will be on the meridian at S_4, and its hour angle will be zero again. *The hour angle is usually reckoned in hours* from 1 to 24, 1 hour being equal to 15 degrees. Thus, when the star is at S_0 its hour angle is *12 h.* The hour angle of a star is partly *local;* for only places on the same meridian of longitude have the same celestial meridian. Moreover, the hour angle of a star is continually changing, and its magnitude depends upon the time of observation. In Arts. 76, 77, the positions of stars are described in terms which are independent of the time and place of observation.

In Arts. 73, 74, 75, the astronomical ideas so far obtained, are used in the solution of two simple problems.

72. The altitude of the pole is equal to the latitude of the place of observation. This theorem, which is necessary in Arts. 73, 74, is the fundamental and most important theorem of spherical astronomy.

In Fig. 55, C represents the centre of the earth, P its north pole, and EQ its equator; O is the place of observation, say some place in the northern hemisphere, Z is its zenith and HR its horizon; CPP_1 is the celestial polar axis, P_1 being the north celestial pole. Draw OP_2 parallel to CP_1, P_2 being on the celestial sphere. The angle ROP_2 is the altitude of the pole at O, since (see Arts. 68, 70) P_1 and P_2 are in the same direction from O.

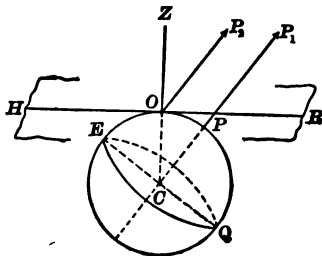



FIG. 55

The latitude of a place is equal to the angle between the plumb line and the plane of the equator. Thus, the latitude of O is equal to OCE. Since OR and OP_2 are respectively perpendicular to CZ and CE, the angle $ROP_2 = OCE$; that is, *the altitude of the pole as observed at O is equal to the latitude of O.*

73. The time of day *can be determined at any place whose latitude is known, if the declination and the altitude of the sun at that time and place are also known.*

NOTE 1. The sun, unlike the stars, changes in declination from 23½° south (about Dec. 22) to 23½° north (about June 21), and then returns south. Its declination is zero, that is, it is on the celestial equator, about March 20 and Sept. 22. This change in declination is due to the revolution of the earth about the sun, and to the fact that the plane of the earth's equator is inclined about 23½° to the plane of its orbit about the sun. The latter plane is called the plane of *the ecliptic.* The declination of the sun is given for each day of the year in the Nautical Almanac. The altitude of the sun can be observed with a sextant.

NOTE 2. The student should consult a text-book on astronomy for an account of the special precautions and corrections necessary in connection with this and similar astronomical problems.

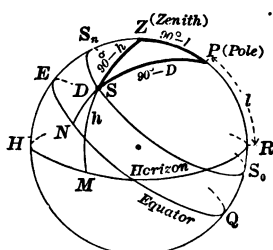



FIG. 56

In Fig. 56, P is the north celestial pole, EQ the celestial equator, S the sun, and S_0SS_n is the small circle on which the sun is moving at the given time; Z is the zenith, and HR the horizon, of the place of observation; ZSM is the sun's vertical circle, and PSN is its hour circle.

It is midnight when the sun is at S_0, and noon when the sun is at S_n. From noon to noon is 24 hours. Hence, *to find the time when the sun is at S, determine the angle ZPS in hours* ($15° = 1$ h.); *subtract* the number of hours from 12, if it is forenoon; and *add,* if it is afternoon.

Let l, h, D, respectively, denote the latitude of the place, and the altitude and declination of the sun.

Then $\quad PR = l$ (Art. 72), $SM = h$, $SN = D$.

In ZPS, whose vertices are the *sun, zenith,* and *pole,*

$$ZP = 90° - l,\quad ZS = 90° - h,\quad SP = 90° - D.$$

Hence, the angle ZPS can be found.

EXAMPLES.

1. In New York (lat. 40° 43′ N.) the sun's altitude is observed to be 30° 40′. What is the time of day, given that the sun's declination is 10° N., and the observation is made in the forenoon?

2. In Montreal (lat. 45° 30′ N.) at an afternoon observation the sun's altitude is 26° 30′. Find the time of day, given that the sun's declination is 8° S.

3. In London (lat. 51° 30′ 48″ N.) at an afternoon observation the sun's altitude is 15° 40′. Find the time of day, given that the sun's declination is 12° S.

4. As in Ex. 2, given that the sun's declination is 18° N.

5. As in Ex. 3, given that the sun's declination is 22° N.

6. As in Ex. 1, given that the sun's declination is 10° S.

74. To find the time of sunrise *at any place whose latitude is known, when the sun's declination is also known.* This is a special case of the preceding problem; for at sunrise the sun is on the horizon and its altitude is zero. The problem can also be solved by means of the triangle RPS_1 (instead of ZPS_1, which is employed in Art 73). For, in RPS_1

$$S_1P = 90° - D,\ PR = l,\ PRS_1 = 90°.$$

$$\therefore\ \cos RPS_1 = \frac{\tan PR}{\tan PS_1} = \frac{\tan l}{\cot D}$$

$$= \tan l \tan D.$$

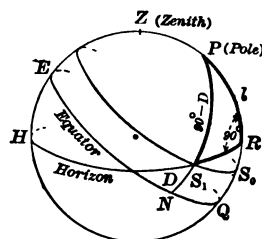



FIG. 57

The angle RPS_1 (*i.e.* S_0PS_1) reduced to hours, gives the time of sunrise (after midnight). If ZPS_1 is found, then ZPS_1 reduced to hours and subtracted from 12 (noon), gives the time of sunrise.. The time of sunset is about as many hours *after* noon as the time of sunrise is before it.

In Fig. 57 the sun is north of the equator. When the sun is south of the equator, $PS_1 = 90° + D$, and $RPS_1 > 90°$ for places in the northern hemisphere. The student can make the figure and investigate this case, and also the case in which the place is in the southern hemisphere.

EXAMPLES.

Find the approximate time of sunrise at a place in latitude l, when the sun's declination is D, in the following cases:

1. $l = 40° 43'$ N. (latitude of New York), D equal to: (*a*) 4° 30′ N. (about April 1); (*b*) 15° 10′ N. (about May 1); (*c*) 23° N. (about June 10); (*d*) 5° N. (about Sept. 10); (*e*) 6° S. (about Oct. 8); (*f*) 15° S. (about Nov. 3); (*g*) 23° S.

2. $l = 51° 30' 48''$ N. (latitude of London), D as in Ex. 1.

3. $l = 60°$ N. (latitude of St. Petersburg), D as in Ex. 1.

4. $l = 70° 40' 7''$ N. (latitude of Hammerfest, Norway, D as in Ex. 1.

5. $l = 29° 58'$ N. (latitude of New Orleans), D as in Ex. 1.

6. $l = 33° 52'$ S. (latitude of Sydney, N. S. W.), D as in Ex. 1.

7. Find the approximate time of sunrise for other days and places.

75. Theorem. *If the latitude of the place of observation is known, then the declination and hour angle of a star can be determined from its altitude and azimuth, and vice versa.* For, in the triangle ZPS (Fig. 56), $ZP = 90° - l$, $SP = 90° - D$, $SZ = 90° - h$, $SPZ = 360° - H.A.$, $PZS = A - 180°$. Hence, if the latitude and any two of the four quantities, viz., altitude, azimuth, declination, hour angle, be known, then the remaining two can be found by solving the triangle SPZ.

76. The equator system: Positions described by declination and right ascension. In the system in Art. 71 the circles of reference were the equator and the meridian of the observer. In the system in this article the circles of reference are *the equator and the circle passing through the celestial poles and the vernal equinox.* **The vernal equinox** is one of the points where the ecliptic intersects the equator; namely, the point where the sun, in its (apparent) yearly path among the stars, crosses the equator in spring. (See text-book on astronomy.) This point may be called *the Greenwich of the celestial sphere.* (*The ecliptic* is the projection of the plane of the earth's orbit on the celestial sphere. The plane of the equator and the plane of the ecliptic are inclined to each other at an angle of $23\frac{1}{2}°$. See Art. 73, Note 1.)

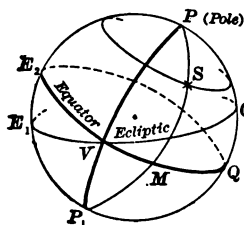



FIG. 58

The right ascension (denoted by R.A.) of a heavenly body is the angle at the north celestial pole between the hour circle of the body and the hour circle of the vernal equinox. This angle is measured from the latter circle *towards the east*, from 0° to 360° or 1 h. to 24 h.; it may be measured by the arc intercepted on the equator. Declination has been defined in Art. 71.

In Fig. 58, P is the north celestial pole, E_2Q the equator, E_1C the ecliptic, and V the vernal equinox. If S is any star, then for S $\quad D = SM$, and R.A. = angle VPM = arc VM.

77. The ecliptic system: Positions described by latitude and longitude. In this system the point and circles of reference are *the pole of the ecliptic, the ecliptic,* and *the great circle passing through the pole of the ecliptic and the vernal equinox.* The **latitude** of a star is its angular (or arcual) distance from the ecliptic; its **longitude** is the angle at the pole of the ecliptic between the circle passing through this pole and the vernal equinox and the circle passing through this pole and the star. This angle may be measured by the arc intercepted on the ecliptic. It is always measured *towards the east* from the vernal equinox.

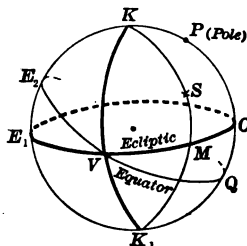



FIG. 59

In Fig. 59, K is the pole of the ecliptic, E_1C the ecliptic, P the pole of the equator, E_2Q the equator, and V the vernal equinox. If S is any star, then

$$\text{latitude of } S = SM, \text{ longitude of } S = VKM = VM.$$

When the latitude and longitude of a star are known, its declination and right ascension can be found, and vice versa. For, in the triangle KPS (the triangle whose vertices are the star and the poles of the equator and the ecliptic), $KP = 23\frac{1}{2}°$ (since $QVC = 23\frac{1}{2}°$), $KS = 90° -$ lat., $SKP = 90° -$ long., $SP = 90° - D$; $SPK = VPK - VPS = 90° - (360° - \text{R.A.})$, if S is west of VP; $SPK = 90° +$ R.A., if S is east of VP. If any two of these be known besides KP, the remaining two can be found by solving KPS.

N.B. *Questions and exercises on Chapter VII. will be found at page* 109.

APPENDIX.

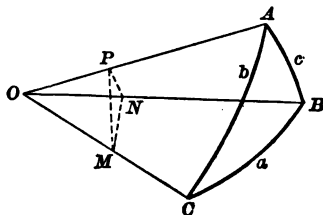
NOTE A.

ON THE FUNDAMENTAL FORMULAS OF SPHERICAL TRIGONOMETRY.

1. The relations between the sides and angles of a right-angled spherical triangle were obtained in Art. 26. The law of sines and the law of cosines (Art. 36) for any spherical triangle have been derived by means of these relations. (See Note 1, Art. 36.) These two laws can also be derived directly by geometry; this is done in Arts. 2, 3, below. Moreover, the law of sines can be derived analytically from the law of cosines, as shown in Art. 4. In Art. 5 it is shown how the relations for right-angled triangles can be derived from these two laws. Other relations between the parts of a spherical triangle have been referred to in Art. 40; these relations can also be deduced by means of the law of cosines and the law of sines. *The law of cosines is, accordingly, the fundamental and most important formula in spherical trigonometry.*

2. Direct geometrical derivation of the law of cosines. Let O–ABC be a triedral angle, and ABC be the corresponding spherical triangle on a sphere of radius OA. It is required to find the cosine of the face angle COB, or, what is the same thing, the cosine of the side CB.

FIG. 60

In OA take any point P, and through P pass a plane MPN at right angles to the line OA. Then OPN and OPM are right angles, and angle $MPN =$ angle A. Also, the measures (in degrees) of the sides AB, BC, CA, are the same as the measures of the face angles COB, BOA, AOC, respectively.

In MPN, $\quad \overline{MN}^2 = \overline{MP}^2 + \overline{PN}^2 - 2\,MP \cdot PN \cos MPN;$ **(1)**

In MON, $\quad \overline{MN}^2 = \overline{MO}^2 + \overline{ON}^2 - 2\,MO \cdot ON \cos MON.$ (2)

Hence, on equating these values of $\overline{MN}^2$ and transposing,

$$2\,MO \cdot ON \cos MON = \overline{MO}^2 - \overline{MP}^2 + \overline{ON}^2 - \overline{PN}^2 + 2\,MP \cdot PN \cos MPN.$$

Now $\overline{OM}^2 - \overline{MP}^2 = \overline{OP}^2$, and $\overline{ON}^2 - \overline{PN}^2 = \overline{OP}^2$, since OPM and OPN are right angles.

$$\therefore\ 2\,MO \cdot ON \cos MON = 2\,OP^2 + 2\,MP \cdot PN \cos MPN.$$

$$\therefore\ \cos MON = \frac{OP}{MO}\,\frac{OP}{ON} + \frac{MP}{MO}\,\frac{PN}{ON} \cos MPN;$$

i.e. $$\cos a = \cos b \cos c + \sin b \sin c \cos A. \qquad (3)$$

Like formulas for $\cos b$, $\cos c$, can be derived in a similar manner; they can also be written immediately, on paying regard to the symmetry in (3). The formulas for $\cos A$, $\cos B$, and $\cos C$, can be derived by means of the polar triangle, as done in Art. 36, *C*.

EXERCISES.

1. Make the figure and derive the law of cosines: (*a*) when P is taken at A; (*b*) when P is taken in OA produced towards A.

2. Derive the formula for *cos b* geometrically. (Take any point in OB, and through this point pass a plane at right angles to OB.)

3. Derive the formula for *cos c* geometrically

3. Direct geometrical derivation of the law of sines. Let $O\text{–}ABC$ be a triedral angle, and ABC be the corresponding spherical triangle on a sphere of radius OA.

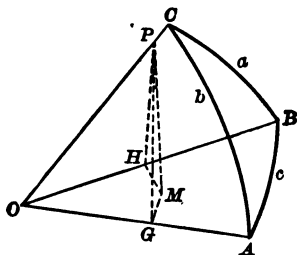



FIG. 61

In OC take any point P, and draw PM at right angles to the plane AOB, and intersecting this plane in M. Through M draw MG and MH, at right angles to OA and OB respectively. Pass a plane through the lines PM and MG.

Since PM is perpendicular to OAB, the plane PMG is perpendicular to OAB (Euc. XI. 18). Hence, since AGM is a right angle, AGP is also a right angle. Therefore angle PGM = angle A. Similarly it can be shown that angle PHM = angle B.

$$\therefore\ \sin A = \frac{PM}{PG} = \frac{PM}{OP \sin AOC} = \frac{PM}{OP \sin b}. \quad \therefore\ \sin A \sin b = \frac{PM}{OP}. \qquad (1)$$

Also, $\sin B = \frac{PM}{PH} = \frac{PM}{OP \sin BOC} = \frac{PM}{OP \sin a}$. $\therefore \sin B \sin a = \frac{PM}{OP}$. (2)

$\therefore$ by (1), (2), $\sin A \sin b = \sin B \sin a$.

$$\therefore \quad \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b}.$$

In a similar way it can be shown that $\frac{\sin A}{\sin a} = \frac{\sin C}{\sin c}$. Hence

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Ex. 1. Show geometrically:

(*a*) that $\frac{\sin A}{\sin a} = \frac{\sin C}{\sin c}$; (*b*) that $\frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$.

Ex. 2. Make the derivation when M is not in the sector AOB.

4. Analytical derivation of the law of sines from the law of cosines.

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \qquad \text{[From (3) Art. 2]}$$

$$\therefore 1 - \cos^2 A = 1 - \left(\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right)^2$$

$$= \frac{\sin^2 b \sin^2 c - \cos^2 a - \cos^2 b \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c};$$

$$= \frac{(1-\cos^2 b)(1-\cos^2 c) - \cos^2 a - \cos^2 b \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c};$$

i.e. $$\sin^2 A = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c}.$$

$$\therefore \frac{\sin^2 A}{\sin^2 a} = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c}. \qquad (1)$$

Similarly, $\frac{\sin^2 B}{\sin^2 b}$ and $\frac{\sin^2 C}{\sin^2 c}$ can each be shown to be equal to the second member of (1). Hence,

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{2n}{\sin a \sin b \sin c}; \qquad (2)$$

in which $2n$ denotes the positive square root of the numerator of the second member of (1).

Ex. 1. Show the truth of the statement made above.

Ex. 2. Show that the numerator in the second member of (1) is equal to $4 \sin s \sin (s - a) \sin (s - b) \sin (s - c)$.

SUGGESTION. $\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$, and Art. 37, (4).

5. Formulas for right-angled triangles derived from the general formulas.
In the triangle ABC let angle $C = 90°$. Then $\sin C = 1$, and relations (1), p. 45, become (2) and (2′), p. 30. Also, $\cos C = 0$, and the third formula in Art. 36, B becomes (1), p. 30. The three formulas in Art. 36, C reduce to (5), (5′), and (6), p. 30, respectively. Formulas (3), (3′), (4) and (4′), p. 30, can be derived from the others on that page. For

$$\cos A = \sin B \cos a \text{ [by (5′)]} = \frac{\sin b}{\sin c} \cdot \frac{\cos c}{\cos b} \text{ [by (2′), (1)]} = \frac{\tan b}{\tan c};$$

similarly, $$\cos B = \frac{\tan a}{\tan c}.$$

Also,

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sin B \cos a} \text{ [by (5′)]} = \frac{\sin a}{\sin b \cos a} \text{ [by (2), (2′)]} = \frac{\tan a}{\sin b};$$

similarly, $$\tan B = \frac{\tan b}{\sin a}.$$

Other relations in triangles (see Art. 40) can also be used in the derivation of the formulas for right-angled triangles.

EXERCISES.

1. Deduce the law of cosines: (1) directly, by geometry; (2) by means of the relations in a right-angled triangle.

2. Deduce the law of sines: (1) analytically, from the law of cosines (2) directly, by geometry; (3) by means of the relations in a right-angled triangle.

3. Deduce the ten relations between the sides and angles of a right-angled spherical triangle: (1) by means of the relations between the sides and angles of the general spherical triangle; (2) directly, by geometry.

NOTE B.

[Supplementary to Art. 58.]

DERIVATION OF FORMULAS FOR THE SPHERICAL EXCESS OF A TRIANGLE.

***I.* Cagnoli's Formula.** (*In terms of the sides.*)

$$\begin{aligned}\sin \tfrac{1}{2} E &= \sin \tfrac{1}{2}(A + B + C - 180°) = -\cos \tfrac{1}{2}(A + B + C)\\
&= \sin \tfrac{1}{2}(A + B) \sin \tfrac{1}{2} C - \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2} C\\
&= \frac{\sin \frac{1}{2} C \cos \frac{1}{2} C}{\cos \frac{1}{2} c}[\cos \tfrac{1}{2}(a - b) - \cos \tfrac{1}{2}(a + b)] \quad \text{[Art. 39, (1), (3)]}\\
&= \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b \sin C}{\cos \frac{1}{2} c} \quad \text{[Arts. 50 (5), 52 (8), \textit{Plane Trig.}]} \quad (1)\\
&= \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b}{\cos \frac{1}{2} c} \cdot \frac{2n}{\sin a \sin b} \quad \text{[Note A, Art. 4, Eq. (2)]}\\
&= \frac{n}{2 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}.\end{aligned}$$

***II.* Lhuillier's Formula.** (*In terms of the sides.*)

$$\tan \tfrac{1}{2} E = \frac{\sin \frac{1}{2}(A+B+C-180^\circ)}{\cos \frac{1}{2}(A+B+C-180^\circ)}$$

$$= \frac{\sin \frac{1}{2}(A+B) - \sin \frac{1}{2}(180^\circ - C)}{\cos \frac{1}{2}(A+B) + \cos \frac{1}{2}(180^\circ - C)} \qquad \text{[\textit{Plane Trig.}, p. 94]}$$

$$= \frac{\sin \frac{1}{2}(A+B) - \cos \frac{1}{2}C}{\cos \frac{1}{2}(A+B) + \sin \frac{1}{2}C}$$

$$= \frac{\cos \frac{1}{2}(a-b) - \cos \frac{1}{2}c}{\cos \frac{1}{2}(a+b) + \cos \frac{1}{2}c} \cdot \frac{\cos \frac{1}{2}C}{\sin \frac{1}{2}C} \qquad \text{[Art. 39, (1), (3)]}$$

$$= \frac{\sin \frac{1}{2}(s-b) \sin \frac{1}{2}(s-a)}{\cos \frac{1}{2}s \cos \frac{1}{2}(s-c)} \sqrt{\frac{\sin s \sin (s-c)}{\sin (s-a) \sin (s-b)}}$$

[Art. 37, (6); *Plane Trig.*, p. 94]

$$= \sqrt{\tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)}.$$

***III.* Formula in terms of two sides and their included angle.**

$$\cos \tfrac{1}{2} E = \cos \tfrac{1}{2}(A+B+C-180^\circ) = \sin \tfrac{1}{2}(A+B+C)$$

$$= \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}C + \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}C$$

$$= [\cos \tfrac{1}{2}(a+b) \sin^2 \tfrac{1}{2}C + \cos \tfrac{1}{2}(a-b) \cos^2 \tfrac{1}{2}C] \sec \tfrac{1}{2}c$$

[Art. 39, (1), (3)]

$$= (\cos \tfrac{1}{2}a \cos \tfrac{1}{2}b + \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \cos C) \sec \tfrac{1}{2}c. \qquad (2)$$

Hence, from (1) and (2), on division and reduction,

$$\tan \tfrac{1}{2} E = \frac{\tan \frac{1}{2}a \tan \frac{1}{2}b \sin C}{1 + \tan \frac{1}{2}a \tan \frac{1}{2}b \cos C}.$$

On taking the reciprocals and reducing, this takes the form

$$\cot \tfrac{1}{2} E = \frac{\cot \frac{1}{2}a \cot \frac{1}{2}b + \cos C}{\sin C}.$$

QUESTIONS AND EXERCISES FOR PRACTICE AND REVIEW.

CHAPTER I.

1. On a sphere let N be the pole of a great circle ABC, and P be any point on the surface between N and ABC; also let $DPNG$ be a semicircle drawn through P at right angles to ABC, and let it intersect ABC in D and G: prove (*a*) that PD is the shortest great-circle arc that can be drawn from P to ABC; (*b*) that PNG is the longest great-circle arc that can be drawn from P to ABC.

2. Show that the greater the distance of the plane of a small circle from the centre of the sphere, the less is the circle.

3. The radius of a sphere is 10 inches, and the radius of a small circle upon it is 6 inches. Find: (*a*) the distance between the centre of the sphere and the centre of the small circle; (*b*) the angular radius of the small circle; (*c*) the polar distance (or arcual radius) of the small circle; (*d*) the distance on the sphere from the small circle to the great circle having the same axis.

4. Prove that if a spherical triangle has two right angles, the sides opposite them are quadrants, and the third angle has the same measure as its opposite side.

5. Prove that in any spherical right triangle an angle and its opposite side are always in the same quadrant.

6. Prove that any side of a spherical triangle is greater than the difference between the other two sides.

7. Prove that each angle of a spherical triangle is greater than the difference between 180° and the sum of the other two angles.

8. Show that the surface of a sphere is eight times the surface of a trirectangular triangle.

9. (*a*) Show that a trirectangular triangle is its own polar; (*b*) show that a triquadrantal triangle is its own polar.

10. Show that if two great circles are equally inclined to a third, their poles are equidistant from the pole of the third.

11. Show that the arc through the poles of two great circles cuts both circles at right angles.

12. A ship sails along the parallel of 45° N. a distance of 600 nautical miles. Find the difference of longitude that she has made.

13. Two places in latitude 60° N. are 150 statute miles apart. Find their difference of longitude. [Take the radius of the earth as 3960 miles.]

14. Compare the lengths of the parallels of 30° N., 45° N., and 60° N., with the length of the equator.

15. Prove that if the first of two spherical triangles is the polar triangle of the second, then the second is the polar triangle of the first.

16. Show that in two polar triangles each angle of the one is the supplement of the side opposite to it in the other.

17. Show that the sum of the angles of a spherical triangle is greater than two, and less than six, right angles.

18. Discuss the following cases, in which A, a, and b are given in a spherical triangle ABC:

I. $A=90°$: (1) $b=90°$; (2) $b<90°(a<b,\ a=b,\ a>b$ and $<\pi-b,\ a=\pi-b,\ a>\pi-b)$; (3) $b>90°(a<\pi-b,\ a=\pi-b,\ a>\pi-b$ and $<b,\ a=b,\ a>b)$.

II. $A<90°$: (1) $b=90°(a<A,\ a=A,\ a>A$ and $<b,\ a=b=90°,\ a>b)$; (2) $b<90°(a<p,\ a=p,\ a>p$ and $<b,\ a=b,\ a>b$ and $<\pi-b,\ a=\pi-b,\ a>\pi-b)$; (3) $b>90°(a<p,\ a=p,\ a>p$ and $<\pi-b,\ a=\pi-b,\ a>\pi-b$ and $<b,\ a=b,\ a>b)$. [For definition of p, see p. 26.]

III. $A>90°$: (1) $b=90°(a=b$, a between b and $\pi-b$, a between $\pi-b$ and $p)$; (2) $b<90°(a>p,\ a=p,\ a<p$ and $>b,\ a=b$, a between b and $\pi-b)$; (3) $b>90°(a<b,\ a>b$ and $<p,\ a<p$ and $>\pi-b$, a between b and $\pi-b,\ a=b)$.

CHAPTER II.

1. Define *spherical angle, spherical triangle, Napier's circular parts, polar triangle, quadrantal triangle, oblique spherical triangle, pole of an arc, spherical excess, spherical polygon.*

2. In a right-angled spherical triangle show that: (a) It is impossible for only one of the three sides to be greater than 90°; (b) The hypotenuse is less than 90° only when both the other sides are in the same quadrant; (c) If another part besides the right angle be right, the triangle is biquadrantal.

3. Prove, by geometry and by trigonometry, that in a right spherical triangle an angle and its opposite side are always in the same quadrant, that is, either both are less or both are greater than 90°.

4. Prove that in a right spherical triangle ABC, $(C = 90°)$. (*a*) $\sin A = \cos B \div \cos b$; (*b*) $\cos c = \cot A \cot B$; (*c*) $\cos c = \cos a \cos b$.

5. (*a*) Mention in order Napier's circular parts, and state the two principal rules for their use. (*b*) State Napier's Rules and write the ten formulas for the right spherical triangle by means of them. (*c*) Prove three of these formulas.

6. What formulas should be used to find B, a, and b of a right spherical triangle ABC $(C = 90°)$ when A and c are given? What formula includes all the required parts?

7. Show how to obtain the formulas for finding a, B, and C of a quadrantal triangle, when A and b are given and $c = 90°$.

8. Given one side and the hypotenuse of a right spherical triangle, write all the formulas for the solution and check, and state how the species of each part will be determined.

9. How many solutions are there for a right spherical triangle ABC, given side b and angle B? Discuss fully.

10. Given A and b of a right spherical triangle ABC $(C = 90°)$: write and derive formulas for computing each of the parts B, a, and c in terms of A and b only; also the check formula.

11. Show how to solve a right spherical triangle, having given (*a*) the sides about the right angle; (*b*) the two oblique angles.

12. (*a*) Show how the solution of a quadrantal triangle may be reduced to that of a right triangle. (*b*) Write the relations between the sides and angles of a quadrantal triangle ABC, in which $c = 90°$.

13. In a spherical triangle ABC, $A = B$: write the relations between the sides and angles of ABC.

14. If A be one of the base angles of an isosceles spherical triangle whose vertical angle is $90°$ and a the opposite side, prove that $\cos a = \cot A$; and determine the limits within which it is necessary that A must lie.

15. Show how oblique spherical triangles can be solved by means of right spherical triangles. (Six cases.)

16. In a right spherical triangle ABC $(C = 90°)$ prove that: (*a*) $\sin^2 B - \cos^2 A = \sin^2 b \sin^2 A$; (*b*) $\sin A \sin 2b = \sin c \sin 2B$; (*c*) $\sin 2A \sin c = \sin 2a \sin B$; (*d*) $\sin 2a \sin 2b = 4 \cos A \cos B \sin^2 c$; (*e*) $\cos^2 A \sin^2 c = \sin^2 c - \sin^2 a$; (*f*) $\sin^2 A \cos^2 c = \sin^2 A - \sin^2 a$.

17. (*a*) In ABC, if $C = 90°$, and $a = b = c$, prove that $\sec A = 1 + \sec a$. (*b*) In ABC $(C = 90°)$ show that if $b = c = \frac{\pi}{2}$, then $\cos a = \cos A$.

18. In a right spherical triangle whose oblique angles are $72° 34'$ and $59° 42'$, find the length of the perpendicular from the right angle upon the base, and the angles which it forms with the sides.

19. Two planes intersecting at right angles are intersected by a third plane making with them angles of 60° and 75° respectively. Find the angles which the three lines of intersection make with each other.

20. Two planes intersect at right angles; from any point of their line of intersection one line is drawn in each plane making the respective angles 60° and 73° with the line of intersection. Find the angle between the two lines thus drawn.

21. A triangle whose sides are 40°, 90°, and 125° respectively, is drawn on the surface of a sphere whose radius is 8 feet. Find in feet the length of each side of this triangle, and also the angles of the polar triangle. Write the formula for finding either angle in terms of functions of the sides.

22. Solve the following spherical triangles given: (1) Right triangle, hypotenuse = 140°, one side = 20°. (2) Sides 90°, 50°, 50°. (3) Sides 100°, 50°, 60°. (4) Sides each 30° in length. (5) $A = 100°$, $C = 90°$, $a = 112°$. (6) $A = 80°$, $a = 90°$, $b = 37°$. (7) $a = b = 119°$, $C = 85°$. (8) Triangle PQR, $R = 90°$, $P = 63° 42'$, $Q = 123° 18'$. (9) Right triangle, one angle = 110° 30′ 20″, hypotenuse = 75° 45′. (10) $A = 90°$, $b = 21° 30'$, $c = 122° 18'$. (11) $B = 90°$, $C = 79° 40'$, $b = 137° 52'$. (12) $A = 90°$, $a = 108° 23$, $c = 37° 42'$. (13) $B = 90°$, $A = 43° 10'$, $a = 78° 35'$. (14) $B = 90°$, $C = 33° 57'$, $A = 43° 18'$. (15) $A = 87° 40' 20''$, $b = 33° 42' 40''$, $B = 90°$. (16) $A = 33° 42' 40''$, $b = 87° 40' 20''$, $B = 90°$.

CHAPTER III.

1. In a spherical triangle ABC prove that: (*a*) $\sin a : \sin A = \sin b : \sin B = \sin c : \sin C$; (*b*) $\cos a = \cos b \cos c + \sin b \sin c \cos A$; (*c*) $\cos A = -\cos B \cos C + \sin B \sin C \cos a$; (*d*) $\cos \frac{1}{2} A = \sqrt{\sin s \sin(s-a) \div \sin b \sin c}$, where $s = \frac{1}{2}(a + b + c)$; (*e*) $\tan \frac{1}{2} A \cot \frac{1}{2} B = \sin(s - b) \operatorname{cosec}(s - a)$.

2. Give the equations (or proportions) known as Napier's Analogies. Derive them.

3. Derive formulas giving the values of $\sin A$, $\cos A$, $\tan A$, and $\cos c$, in terms of functions of a, b, and c.

4. In a spherical triangle ABC show that: (*a*) If $a = b = c$, then $\sec A = 1 + \sec a$. (*b*) If $b + c = 180°$, then $\sin 2B + \sin 2C = 0$. (*c*) If $C = 90°$, then $\tan \frac{1}{2}(c + a) \tan \frac{1}{2}(c - a) = \tan^2 \frac{b}{2}$.

5. In an equilateral spherical triangle show that: (*a*) $2 \sin \frac{A}{2} \cos \frac{a}{2} = 1$, and hence, that such a triangle can never have its angle less than 60°, nor its side greater than 120°; (*b*) $2 \cos A = 1 - \tan^2 \frac{a}{2}$.

6. Show that: (*a*) If the three angles of spherical triangle ABC are together equal to four right angles, then $\cos^2 \frac{c}{2} = \cot A \cot B$. (*b*) If x is

the side of a spherical triangle formed by joining the middle points of the equilateral triangle of side a, then $2 \sin \frac{x}{2} = \tan \frac{a}{2}$.

7. (*a*) In a spherical triangle ABC show that, if $b + c = 90°$, then $\cos a = \sin 2c \cos^2 \frac{A}{2}$. (*b*) If a be the side of an equilateral triangle and a' that of its polar triangle, prove $\cos a \cos a' = \frac{1}{2}$.

8. (*a*) If, in a triangle ABC, l be the length of the arc joining the middle point of the side c to the opposite vertex C, show that $\cos l = (\cos a + \cos b) \div 2 \cos \frac{c}{2}$. (*b*) In a right spherical triangle ABC ($C = 90°$), if α, β be the arcs drawn from C respectively perpendicular to and bisecting the hypotenuse c, show that $\sin^2 \frac{c}{2}(1 + \sin^2 \alpha) = \sin^2 \beta$.

9. (*a*) Prove that the half sum of two sides of any spherical triangle is in the same quadrant as the half sum of the opposite angles. (*b*) Two sides of a spherical triangle are given: prove that the angle opposite the smaller of them will be greatest when that opposite the larger is a right angle.

10. ABC is a spherical triangle of which each side is a quadrant, and P is a point within it. Prove that $\cos^2 PA + \cos^2 PB + \cos^2 PC = 1$.

11. In a spherical triangle, if $A = 36°$, $B = 60°$, and $C = 90°$, show that $a + b + c = 90°$.

CHAPTER IV.

1. (*a*) Name the six cases for solution of spherical triangles. (*b*) Discuss each case in detail, writing the formulas used in the solution, and deriving these formulas. (*c*) Solve an example under each case. Test the result by (1) solving by right triangles, (2) solving without logarithms, (3) using a check formula.

2. How many solutions are possible for the oblique spherical triangle ABC, given A, B, and a? Discuss in full the question of one solution, two solutions, or no solution. Plan the solution.

3. In a spherical triangle ABC, two sides a and b and the included angle C are given. Write all the formulas used in the solution and check; describe fully the process of solution. Derive the formulas used.

4. Write and deduce the formulas for finding A, B, and C of any spherical triangle when a, b, and c are given.

5. Given A, B, and C. Show how to find the remaining parts, writing the formulas to be used.

6. In an equilateral spherical triangle the side a is given. Find the angle A.

7. Solve the spherical triangle whose sides are 70°, 60°, and 50°. Solve the plane triangle obtained by connecting by straight lines the vertices of this spherical triangle, the sphere on which it is drawn being 2 feet in diameter.

8. In a triangle ABC on the earth's surface (supposed spherical) $a = 483$ miles, $b = 321$ miles, $C = 38° 21'$. Find the length of the side c. [Earth's radius = 3960 miles.]

9. Two planes intersect at an angle of 75°. From any point of their line of intersection one line is drawn in each plane, making the respective angles 55° and 80° with the line of intersection. Find the angle between the lines thus drawn.

10. Two planes intersecting at an angle of 65° are intersected by a third plane, making with them the respective angles 55° and 82°. Find the angles which the three lines of intersection make with one another.

11. A solid angle is contained by three plane angles 62°, 83°, 38°. Find the angle between the planes of the angles 62° and 38°.

12. Two of the three angles which contain a solid angle are 42° and 65° 30′, and their planes are inclined at an angle of 50°. Find the angle of the third plane face and the angles at which this third plane is inclined to the other two planes.

13. A pyramid has each of its slant sides and base an equilateral triangle. Find the angle between any two faces.

14. A pyramid each of whose slant faces is an equilateral triangle has a square base. Find the angle between any two slant faces, also the angle between any slant face and the base.

15. In the following cases ABC is a three-sided spherical figure each of whose sides is an arc of a great circle. Select those which are spherical triangles, and give reasons for so doing. Explain why the other figures cannot be triangles. Solve the triangles and check the results. (Solve some without using logarithms.)

(1) $a = 76°$, $b = 54°$, $c = 36°$. (2) $A = 54° 35' 20''$, $b = 104° 25' 45''$, $c = 92° 10'$. (3) $A = 107° 47' 7''$, $B = 38° 58' 27''$, $c = 51° 41' 14''$. (4) $A = 60°$, $B = 80°$, $C = 100°$. (5) $A = 120°$, $B = 130°$, $C = 80°$. (6) $A = 54° 35'$, $b = 104° 24'$, $c = 95° 10'$. (7) $A = 61° 37' 53''$, $B = 139° 54' 34''$, $b = 150° 17' 26''$. (8) $a = 72° 18'$, $b = 146° 35'$, $c = 98° 11'$. (9) $A = 125° 15'$, $C = 85° 12'$, $b = 100°$. (10) $A = 50°$, $B = 114° 5' 8''$, $b = 50°$. (11) $A = 83° 40'$, $b = 73° 45'$, $a = 30° 24'$. (12) $A = 83° 40'$, $b = 30° 24'$, $a = 73° 45'$. (13) $A = 97° 20'$, $a = 94° 37'$, $b = 36° 17'$. (14) $a = 127° 40'$, $b = 143° 50'$, $c = 139° 39'$. (15) $A = 40°$, $B = 30°$, $C = 20°$. (16) $A = 40° 35'$, $B = 36° 42'$, $c = 47° 18'$.

CHAPTER V.

[In the following exercises,

$$n = \sqrt{\sin s \sin(s-a)\sin(s-b)\sin(s-c)},$$

and $$N = \sqrt{-\cos S \cos(S-A)\cos(S-B)\cos(S-C)};$$

also, r, r_a, r_b, r_c, denote the radii of the circles inscribed in the spherical triangle ABC and its three colunar triangles, and R, R_a, R_b, R_c denote the radii of the circumscribing circles of these triangles.]

1. Given a spherical triangle ABC, find (1) the radius of the inscribed circle; (2) the radius of the circumscribing circle; (3) the radii of the inscribed circles of the colunar triangles; (4) the radii of the circumscribing circles of the colunar triangles.

Show that:

2. $\text{Tan}\, r = \dfrac{n}{\sin s}.$

3. $\text{Tan}\, R = \dfrac{N}{\cos(S-A)\cos(S-B)\cos(S-C)}.$

4. (a) $\text{Cot}\, R \cot R_a \cot R_b \cot R_c = N^2$;

(b) $\text{Tan}\, R \cot R_a \cot R_b \cot R_c = \cos^2 S.$

5. $\text{Tan}\, R = 4 \tan r \dfrac{-\cos S \sin s}{\sin a \sin b \sin c \sin A \sin B \sin C}.$

6. $\text{Tan}\, r_a \tan r_b \tan r_c = \tan r \sin^2 s.$

7. $\text{Tan}\, R + \cot r = \tan R_a + \cot r_a = \tan R_b + \cot r_b$

$= \tan R_c + \cot r_c = \frac{1}{2}(\cot r + \cot r_a + \cot r_b + \cot r_c).$

8. $\text{Tan}\, R \tan r = -\dfrac{\cos S \sin a}{\sin s \sin A} = -\dfrac{\cos S \sin b}{\sin s \sin B} = \text{etc.}$ Write the other formula of this set.

9. $\text{Tan}^2 R + \tan^2 R_a + \tan^2 R_b + \tan^2 R_c = \cot^2 r + \cot^2 r_a + \cot^2 r_b + \cot^2 r_c.$

10. $\text{Tan}\, r \tan r_a \tan r_b \tan r_c = n^2$; $\cot r \tan r_a \tan r_b \tan r_c = \sin^2 s.$

11. In any equilateral triangle, $\tan R = 2 \tan r.$

12. $\text{Tan}\, R_a = -\dfrac{\tan \frac{1}{2} a}{\cos S} = \dfrac{\cos(S-A)}{N} = \dfrac{\sin \frac{1}{2} a}{\sin A \sin \frac{1}{2} b \sin \frac{1}{2} c}$

$= \dfrac{2 \sin \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}{n} = \dfrac{1}{2n}[\sin s - \sin(s-a) + \sin(s-b) + \sin(s-c)].$

Write the corresponding formulas for R_b and R_c.

13. $\text{Cot}\, r_a + \cot r_b + \cot r_c - \cot r = 2 \tan R.$

14. Find the radii of the circles connected with some of the triangles in Ex. 15 of the preceding set.

CHAPTER VI.

1. Define the following terms: *zone* of a sphere, *lune*, *spherical degree*, *spherical excess of a triangle*, *spherical excess of a* (non-re-entrant) *polygon*, *spherical excess* of any figure on a sphere, *spherical measure* and *spherical degree measure* of a solid angle, *spherical pyramid*, *spherical sector*, *spherical segment*.

2. Derive the area of the surface of a sphere.

3. Derive the area of a spherical triangle.

4. Discuss fully the measurement of solid angles.

5. Show how to find the spherical excess of a figure on a sphere when the area of the figure is given (in square units).

6. State and deduce *Roy's Rule* for computing the spherical excess of a triangle of known area on the earth's surface.

7. Derive the volumes of a sphere, a spherical pyramid, a spherical sector, and a spherical segment.

8. The area of an equilateral triangle is one-fourth the area of the sphere: find its sides and angles.

9. If the three sides of a spherical triangle measured on the earth's surface be 12, 16, and 18 miles, find the spherical excess.

10. If $a = b$ and $C = \frac{\pi}{2}$, show that $\tan E° = \frac{\sin^2 a}{2 \cos a}$. (In ABC.)

11. If $a = b = 60°$ and $c = 90°$, show that $E = \cos^{-1} \frac{7}{9}$. (In ABC.)

12. If $C = 90°$ in ABC, then $E = 2 \tan^{-1}(\tan \frac{1}{2} a \tan \frac{1}{2} b)$.

13. In a triangle on the earth's surface (assumed spherical), two sides are 483 and 321 miles, and the angle between them is $38° 21'$. Find the area of the triangle in square miles. [Radius of earth = 3960 miles.]

14. The sides of a triangle on the earth's surface (supposed spherical) are 321, 287, and 412 miles; find the area.

15. Prove that in a right triangle ABC ($C = 90°$),

$$\cos \tfrac{1}{2} E = \frac{\cos \frac{1}{2} a \cos \frac{1}{2} b}{\cos \frac{1}{2} c}, \text{ and } \sin \tfrac{1}{2} E = \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b}{\cos \frac{1}{2} c}.$$

16. The spherical excess of a triangle on the earth's surface is $2''.5$. Find its area, the radius of the earth being taken as 3960 miles.

17. Find the fraction of the earth's surface (supposed spherical) contained by great-circle arcs joining London, New York, and Paris. Find the spherical degree measure, and the spherical measure of the angle subtended at the centre of the earth by this part of the earth's surface.

18. Find the spherical excess of some of the triangles in Ex. 15, p. 104. Also find their areas in square inches on spheres of radii, say, 4 inches, 10 inches, 12 inches, 20 inches, a inches.

19. Find the spherical measures and the spherical degree measures of the solid angles corresponding to the triangles taken in Ex. 18.

CHAPTER VII.

1. Given the latitude and longitude of each of two places: show how to find the shortest distance between these places, and the direction of one place from the other.

2. Given the latitudes and longitudes of three places on the earth's surface, and also the radius of the earth: show how to find the area of the spherical triangle formed by arcs of great circles passing through them.

3. Given the sun's altitude and declination and the latitude of a place: show clearly how the time of day may be determined.

4. If d represents the sun's declination, what formulas will be required in order to determine the time of sunrise for a place whose latitude is l?

5. Show what formulas must be used to find the length of a degree of longitude on the earth's surface for a place whose latitude is l, r representing the radius of the earth.

6. The shortest distance d between two places and their latitudes l and l' are known; find their difference of longitude.

7. Given the obliquity of the ecliptic ω, and the sun's longitude λ, show that if α and δ denote his right ascension and declination respectively, then $\tan\alpha = \cos\omega\tan\lambda$, and $\sin\delta = \sin\omega\sin\lambda$.

8. The faces of a regular dodecaedron are regular pentagons, three faces meeting at each vertex. Find the diedral angle at the edge of the solid.

9. The ridges of two gable roofs meet at right angles; each roof is inclined to the horizontal at an angle of 65°. Find the diedral angle between the planes of the two roofs, and the angle their line of intersection makes with the ridge of either roof.

10. What is the direction of a wall in latitude 52° 30′ N. which casts no shadow at 6 A.M. on the longest day of the year?

11. Two ports are in the same parallel of latitude, their common latitude being l, and their difference of longitude 2λ. Show that the saving of distance in sailing from one to the other on the great circle instead of sailing due east or west, is

$$2r\{\lambda\cos l - \sin^{-1}(\sin\lambda\cos l)\},$$

λ being expressed in radian measure, and r being the radius of the earth.

12. If a ship sails from New York (40° 28′ N., 74° 8′ W.) starting due east, and continues her course on an arc of a great circle, what will be her latitude when she reaches the meridian of Greenwich, and in what direction will she then be sailing?

13. Find the distance between New York (40° 28′ N., 74° 8′ W.) and Cape Clear (51° 26′ N., 9° 29′ W.), and the bearing of each from the other. [Radius of earth = 3960 miles.]

14. From Victoria, B.C. (48° 25′ N., 123° 23′ W.), a ship sails on an arc of a great circle for 1250 miles, starting in the direction S. 47° 35′ W. Find its latitude and longitude, taking the length of 1° as 69¼ miles.

15. Two places are both in latitude 50° N., and the difference of their longitudes is 60°. Find the distance between them (*a*) along the parallel of latitude, (*b*) along a straight line, (*c*) along a great circle. [Earth's radius = 3960 miles.]

16. What will be the first course and the shortest (great circle) distance passed over in sailing from a place in latitude 43° N. to another place 80° east of it and in the same latitude? What is the distance between the two places along the parallel? What is the straight-line distance between them?

17. At what hours will the sun rise in London (51° 30′ 48″ N.) and New York (40° 43′ N.) when its declination is respectively 23° N., 20° N., 15° N., 10° N., 5° N., 5° S., 10° S., 15° S., 20° S., 23° S.?

18. When the sun's declination is 18°, find his right ascension and longitude.

19. What is the altitude of the sun above the horizon when its angular distance from the south point is 75° and from the west point is 60°?

20. The right ascension of Sirius is $6^h\ 38^m\ 37^s.6$, and his declination is 16° 31′ 2″ S.; the right ascension of Aldebaran is $4^h\ 27^m\ 25^s.9$, and his declination is 16° 12′ 27″ N. Find the angular distance between these stars.

21. If the sun's declination be 20° 45′ N. and his altitude be 41° 10′ at 3 P.M., find the observer's latitude.

22. What will be the altitude of the sun at 3.30 P.M. in San Francisco (37° 48′ N.), its declination being 15° S.?

23. In Bombay (18° 54′ N.) the altitude of the sun is observed to be 27° 40′. If the sun's declination is 7° S. and the observation is made in the morning, find the hour of the day.

24. Find the latitude and longitude of a star whose right ascension is $4^h\ 40^m$, and declination 57°.

25. Find the distance in degrees between the sun and moon when their right ascensions are respectively $15^h\ 12'$, $4^h\ 45'$, and their declinations are 21° 30′ S., 5° 30′ N.

26. Find the length of the longest day in the year at the following places (the sun's greatest declination being 23° 27′ N.): London (51° 30′ 48″ N.), New York (40° 43′ N.), Montreal (45° 30′ N.), St. Petersburg (60° N.), Hong Kong (22° 17′ N.).

27. Find the length of the shortest day in the year at the places mentioned in Ex. 26. (The sun's declination is then 23° 27′ S.)

28. At Copenhagen (55° 40′ N.), at an afternoon observation, the sun's altitude is 44° 20′; find the time of day, the sun's declination being 18° 25′ N.

29. At what time of day will the sun have an altitude of 53° 40′ for a place in latitude 40° 35′ N., his declination being 13° 48′ N.?

30. What will be the sun's altitude at 3.30 P.M. at a place in latitude 44° 40′ N., his declination being 18° N.?

31. What will be the sun's altitude at 10 A.M. at a place in latitude 44° 40′ N., his declination being 18° S.?

32. What is the sun's declination when his altitude at a place in latitude 37° 48′ N. is 25° at 4 P.M.?

NOTE. The Spherical Trigonometries of M'Clelland and Preston, Casey, and Bowser, contain especially good collections of exercises. See Art. 40.

ANSWERS TO THE EXAMPLES.

CHAPTER I.

Art. 24. **I. 4.** $A = 88° 12.2'$, $B = 74° 34.7'$, $C = 43° 8'$; $A = 118° 33.2'$, $B = 113° 11.2'$, $C = 92° 45'$. **II. 4.** $a = 72° 40.6'$, $b = 67° 45.8'$, $c = 51° 43.1'$; $a = 71° 22.7'$, $b = 108° 37.3'$, $c = 104° 56.7'$. **III. 4.** $A = 63° 56'$, $B = 126° 21.2'$, $c = 77° 3'$; $B = 32° 47.1'$, $C = 62° 30.7'$, $a = 84° 29.5'$. **IV. 5.** $b = 70° 5.7'$, $c = 102° 51.3'$, $A = 68° 35.8'$; $a = 46° 1.5'$, $c = 86° 0.7'$, $B = 122° 55.8'$. **VI. 3.** $B = 59° 40.1'$, $C = 114° 55'$, $c = 96° 31.1'$, and $B = 120° 19.9'$, $C = 27° 49.6'$, $c = 30° 45.4'$; $B = 65° 1.8'$, $C = 97° 16.9'$, $c = 100° 26'$; $C = 110° 43.1'$, $b = 33° 8.6'$, $c = 60° 28.8'$; $C = 165° 3.3'$, $b = 125° 1.7'$, $c = 162° 55.7'$, and $C = 119° 47'$, $c = 81° 7'$, $b = 54° 58.3'$.

CHAPTER II.

Art. 27. **4.** $c = 82° 33.9'$, $A = 60° 51.2'$, $B = 76° 56.1'$. **5.** $a = 33° 0.25'$, $b = 36° 29.4'$, $c = 47° 37.8'$.

Art. 31. **5.** (1) $C = 86° 30.9'$, $A = 36° 30.2'$, $B = 87° 25.4'$. (2) $b = 138° 24.4'$, $A = 58° 41.9'$, $B = 129° 43.1'$. (3) $a = 35° 50.6'$, $b = 75° 39.5'$, $B = 81° 29.1'$. (4) $a = 42° 49.8'$, $b = 27° 47.3'$, $c = 49° 33'$. (5) $b = 33° 37.4'$, $c = 79° 2'$, $B = 34° 20.1'$; and $b = 146° 22.6'$, $c = 100° 58'$, $B = 145° 39.9'$. (6) $a = 35° 16.4'$, $c = 51° 10.8'$, $B = 55° 18.6'$.

Art. 32. **1.** (1) $b = 54° 20'$, $A = 32° 0.75'$, $B = 57° 59.25'$, $C = 93° 59.3'$; (2) $b = 66° 29'$, $c = 111° 29.4'$, $B = 50° 17'$, $C = 128° 41.2'$. **2.** (1) $b = 59° 56.2'$, $A = 130°$, $B = 52° 55.5'$. (2) $a = 135° 33'$, $b = 100° 58.6'$, $C = 101° 24.7'$.

CHAPTER III.

Art. 37. **I. 2.** $A = 55° 58.4'$, $B = 74° 14.6'$, $C = 103° 36.6'$. **3.** $A = 43° 58'$, $B = 58° 14.4'$, $C = 108° 4.8'$. **II. 3.** $a = 39° 29.6'$, $b = 35° 36.2'$, $c = 27° 59'$. **4.** $a = 130° 49.6'$, $b = 120° 17.5'$, $c = 54° 56.1'$.

CHAPTER IV.

Art. 42. **2.** $A = 41° 27'$, $B = 66° 26.4'$, $C = 106° 3.2'$. **3.** $A = 144° 26.6'$, $B = 26° 9.1'$, $C = 36° 34.7'$.

Art. 43. **1.** $a = 43° 36'$, $b = 41° 20.9'$, $c = 33° 7.4'$. **2.** $a = 111° 40.2'$, $b = 91° 17.2'$, $c = 71° 7.4'$.

Art. 44. **2.** $A = 101° 24.2'$, $B = 54° 57.9'$, $c = 79° 9.5'$. **3.** $B = 78° 20.6'$, $C = 47° 47'$, $a = 82° 42'$.

Art. 45. **1.** $a = 63° 15.1'$, $b = 43° 53.7'$, $C = 95° 1'$. **2.** $b = 86° 39.5'$, $c = 68° 39.5'$, $A = 59° 44'$.

Art. 46. **2.** $B = 36° 35.5'$, $C = 51° 59.7'$, $c = 42° 38.9'$. **3.** $B = 59° 3.5'$, $C = 97° 38.8'$, $c = 56° 56.9'$; $B = 120° 56.5'$, $C = 28° 5.2'$, $c = 23° 27.8'$.

Art. 47. **1.** $b = 154° 45.1'$, $c = 34° 9.1'$, $C = 70° 17.5'$. **2.** $A = 164° 43.7'$, $a = 162° 37.5'$, $c = 124° 40.6'$; $A = 119° 18.7'$, $a = 81° 18.7'$, $c = 55° 19.4'$.

CHAPTER VI.

Art. 53. **1.** 2827.44 sq. in. **2.** 392.7 sq. in. **3.** 8.25 sq. ft.

Art. 55. **1.** 1.396 sq. ft. **2.** 64.14 sq. ft.

Art. 56. 24° 37′ 47″ (.42986), 33° 56.6′ (.59213), 27° 10.4′ (.47426), 12° (.20944), 86° 20′ (1.5068), etc.

Art. 57. **1.** 42.986 sq. ft., 59.213 sq. ft., 47.426 sq. ft. **2.** 130.9 sq. in., 941.75 sq. in.

Art. 61. **1.** Spherical degree measure = 12, spherical measure = .20944, **2.** Spherical degree measure = 24.63, spherical measure = .42986.

Art. 64. **1.** 143.29 cu. ft., 197.38 cu. ft., 158.09 cu. ft., 1090.8 cu. in., 7847.9 cu. in., etc. **2.** (*a*) 1357.17 cu. in. (*b*) 904.78 cu. ft.

CHAPTER VII.

Art. 66. **2.** 8° 4.3′ S.; course, S. 45° 6′ E. **5.** (*a*) On the equator in long. 18° 56′ E.; course, S. 47° 39′ E. (*b*) Lat. 42° 21′ S., long. 108° 56′ E.; course, E. (*c*) On the equator in long. 161° 4′ W.; course, N. 47° 39′ E. **6.** Distance = (51° 19.8′) = 3547.675 mi.; bearing of New York from Liverpool is N. 71° 6.8′ W., and bearing of Liverpool from New York is N. 48° 5.8′ E.; lat. 51° 44.1′ N.; course, N. 65° 38′ E.

Art. 73. **1.** 8.08 A.M. **2.** 2.33 P.M. **3.** 2.59 P.M. **4.** 4.09 P.M. **5.** 6.09 P.M. **6.** 9.46 A.M.

Art. 74. **1.** (*a*) 5.44; (*b*) 5.06; (*c*) 4.34; (*d*) 5.43; (*e*) 6.21; (*f*) 6.53; (*g*) 7.26. **2.** (*a*) 5.37; (*b*) 4.40; (*c*) 3.51; (*d*) 5.35; (*e*) 6.30; (*f*) 7.19; (*g*) 8.09. **3.** (*a*) 5.29; (*b*) 4.08; (*c*) 2.51; (*d*) 5.25; (*e*) 6.42; (*f*) 7.51. (*g*) 9.09.